

# A FAMILY OF SYMMETRIC FUNCTIONS ASSOCIATED WITH STIRLING PERMUTATIONS

RAFAEL S. GONZÁLEZ D'LEÓN\*

**ABSTRACT.** We present exponential generating function analogues to two classical identities involving the ordinary generating function of the complete homogeneous symmetric functions. After a suitable specialization the new identities reduce to identities involving the first and second order Eulerian polynomials. The study of these identities led us to consider a family of symmetric functions associated with the *Stirling permutations* introduced by Gesel and Stanley. In particular, we define certain type statistics on Stirling permutations that refine the statistics of descents, ascents and plateaux and we show that their refined versions are equidistributed generalizing a result of Bóna. The definition of this family of symmetric functions extends to the generality of  $r$ -Stirling permutations or  $r$ -multipermutations. We discuss some occurrences of these symmetric functions in the cases of  $r = 1$  and  $r = 2$ .

## 1. INTRODUCTION

We denote by  $\mathbb{N}$  the set of nonnegative integers,  $\mathbb{P}$  the set of positive integers and  $\mathbb{Q}$  the set of rational numbers. Let  $\mathbf{x} = x_1, x_2, \dots$  be an infinite set of variables and  $\Lambda = \Lambda_{\mathbb{Q}}$  the ring of symmetric functions with rational coefficients on the variables  $\mathbf{x}$ , that is, the ring of power series on  $\mathbf{x}$  of bounded degree that are invariant under permutation of the variables. A *weak composition* is an infinite sequence  $\mu = (\mu_1, \mu_2, \dots)$  of numbers  $\mu_i \in \mathbb{N}$  such that its *sum*  $|\mu| := \sum_i \mu_i$  is finite. If  $|\mu| = n$  for some  $n \in \mathbb{N}$ , we say that  $\mu$  is a *weak composition of  $n$* . We denote by  $\text{wcomp}$  the set of weak compositions and  $\text{wcomp}_n$  the set of weak compositions of  $n$ . An *(integer) partition*  $\lambda$  of  $n$  (denoted  $\lambda \vdash n$ ) is a weak composition of  $n$  whose entries are nonincreasing, i.e.,  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ . If  $\nu \in \text{wcomp}$  is obtained by permuting the entries of another  $\mu \in \text{wcomp}$  we say that  $\nu$  is a *rearrangement* of  $\mu$ . We denote the set of rearrangements of  $\mu$  by  $\text{wcomp}_{\mu}$ . Throughout this document we also denote  $x^{\mu} := \prod_i x_i^{\mu_i}$  for  $\mu \in \text{wcomp}$ .

---

\*Partially supported by NSF Grant DMS 1202755.

Define  $h_0 = e_0 := 1$  and for  $n \geq 1$ , the  $n$ th complete homogeneous symmetric function

$$h_n := \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

and the  $n$ th elementary symmetric function

$$e_n := \sum_{1 \leq i_1 < i_2 < \dots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

For  $\lambda \vdash n$  the complete homogeneous and elementary symmetric functions corresponding to  $\lambda$  are defined multiplicatively, i.e.,  $h_\lambda := \prod_i h_{\lambda_i}$  and  $e_\lambda := \prod_i e_{\lambda_i}$  respectively. It is known that for  $n \geq 0$ , the sets  $\{h_\lambda \mid \lambda \vdash n\}$  and  $\{e_\lambda \mid \lambda \vdash n\}$  are bases for the  $n$ -th homogeneous graded component of  $\Lambda_{\mathbb{Q}}$ , where the grading is with respect to degree. For information not presented here regarding symmetric functions the reader could go to [17] and [24, Chapter 7].

For a sequence  $(a_0, a_1, \dots)$  of elements in a ring  $R$  (containing  $\mathbb{Q}$ ) the *ordinary generating function* (or *o.g.f*) of  $(a_n)$  is the formal power series  $\sum_{n \geq 0} a_n y^n \in R[[y]]$  and the *exponential generating function* (or *e.g.f*) of  $(a_n)$  is the formal power series  $\sum_{n \geq 0} a_n \frac{y^n}{n!} \in R[[y]]$  (cf. [25]).

In all the following  $f^{-1}$  denotes the multiplicative inverse and  $f^{\langle -1 \rangle}$  denotes the compositional inverse of  $f \in R[[y]]$  whenever any of these inverses exist. We are going to consider the ring  $\Lambda[[y]]$  of power series in the variable  $y$  with coefficients in  $\Lambda$ . In particular the following two identities are classical results in the study of symmetric functions.

**Proposition 1.1** (cf. [17]). *We have*

$$\left( \sum_{n \geq 0} (-1)^n h_n(\mathbf{x}) y^n \right)^{-1} = \sum_{n \geq 0} e_n(\mathbf{x}) y^n.$$

**Proposition 1.2** (cf. [23]). *We have*

$$\left( \sum_{n \geq 1} (-1)^{n-1} h_{n-1}(\mathbf{x}) y^n \right)^{\langle -1 \rangle} = \sum_{n \geq 1} \omega PF_{n-1}(\mathbf{x}) y^n,$$

where  $\omega$  is the involution in  $\Lambda$  that maps  $h_n(\mathbf{x})$  to  $e_n(\mathbf{x})$  and  $PF_{n-1}(\mathbf{x})$  is Haiman's parking function symmetric function (cf. [13]).

Stanley proved in [23] that  $\omega PF_{n-1}(\mathbf{x})$  has positive coefficients when expressed in the elementary basis, a property known as *e-positivity*.

**Proposition 1.3** ([23]). *For  $n \geq 0$ ,*

$$\omega PF_n(\mathbf{x}) = \sum_{\pi \in \mathbf{NC}_n} e_{\lambda(\pi)}(\mathbf{x}),$$

where  $\mathbf{NC}_n$  is the set of noncrossing partitions of  $[n]$  (see [23]) and  $\lambda(\pi)$  is the integer partition of  $n$  whose parts are the sizes of the blocks of the set partition  $\pi$ .

A common feature of Propositions 1.1 and 1.2 is the e-positivity of the coefficients of the power series in the right-hand side.

In this work we find exponential generating function analogues to Propositions 1.1 and 1.2. We prove the following theorems.

**Theorem 1.4.** *We have*

$$(1.1) \quad \left( \sum_{n \geq 0} (-1)^n h_n(\mathbf{x}) \frac{y^n}{n!} \right)^{-1} = \sum_{n \geq 0} \left( \sum_{\sigma \in \mathfrak{S}_n} e_{\lambda(\sigma)}(\mathbf{x}) \right) \frac{y^n}{n!},$$

where  $\mathfrak{S}_n$  is the set of permutations of  $[n] =: \{1, 2, \dots, n\}$  and  $\lambda(\sigma)$  is the consecutive ascending type of  $\sigma \in \mathfrak{S}_n$  (defined later).

**Theorem 1.5.** *We have*

$$(1.2) \quad \left( \sum_{n \geq 1} (-1)^{n-1} h_{n-1}(\mathbf{x}) \frac{y^n}{n!} \right)^{\langle -1 \rangle} = \sum_{n \geq 1} \left( \sum_{\theta \in \mathcal{Q}_{n-1}} e_{\lambda(\theta)}(\mathbf{x}) \right) \frac{y^n}{n!},$$

where  $\mathcal{Q}_n$  is the set of Stirling (multi)permutations of the multiset  $[n] \sqcup [n] =: \{1, 1, 2, 2, \dots, n, n\}$  (defined by Gessel and Stanley in [7]) and  $\lambda(\theta)$  is any of various types (defined later) for  $\theta \in \mathcal{Q}_n$ .

Theorem 1.5 was derived in [9] using poset topology techniques applied to a poset of partitions weighted by weak compositions. The coefficient of  $\frac{y^n}{n!}$  in the power series of the right-hand side of equation (1.2) is the generating function for the dimensions of the reduced (co)homology of the maximal intervals of the poset of weighted partitions and also for the dimensions of the multilinear components of the free Lie algebra with multiple compatible brackets on  $n$  generators. Here we take a different approach that does not involve poset theoretic techniques. We provide a different combinatorial proof of Theorem 1.5 using Drake's interpretation of the compositional inverse of an exponential generating function in [4].

In order to apply Drake's theorem we study a subset of the set of planar leaf-labeled binary trees that we call *normalized*. The normalization condition means that in any subtree the smallest label is in

the leftmost leaf. This is equivalent to consider non-planar leaf-labeled binary trees (or phylogenetic trees) and the normalization condition is just a particular choice on how to draw these trees in the plane. Using Drake's technique we prove a version of Theorem 1.5 in terms of normalized trees (instead of Stirling permutations) and use this result and a bijection used in [9] between normalized trees and Stirling permutations (a bijection that appeared first in [3]) to derive Theorem 1.5.

We then generalize the symmetric functions that appear as coefficients of the power series of the right-hand side of equation (1.2) to the generality of the family  $\mathcal{Q}_n(r)$  of  $r$ -Stirling permutations, where Stirling permutations correspond to the case  $r = 2$  and the classical permutations in the symmetric group to the case  $r = 1$ . We consider the family of symmetric functions

$$(1.3) \quad SP_n^{(r)}(\mathbf{x}) = \sum_{\theta \in \mathcal{Q}_n(r)} e_{\lambda(\theta)}(\mathbf{x}),$$

where  $\lambda(\theta)$  is any of various types of  $\theta$  (defined later).

It turns out that the case  $r = 1$  is the family of symmetric functions that appear in the right hand side of equation (1.1). In order to prove Theorem 1.4, this time we use an interpretation of the multiplicative inverse of an exponential generating function that can be derived from a more general result discovered by Fröberg [6], Carlitz-Scoville-Vaughan [2] and Gessel [8]. As in the case of Theorem 1.5 there is a similar proof of Theorem 1.4 using poset topology techniques over a poset of subsets weighted by weak compositions. Some of the context of that proof is discussed in Section 6.2.

We note that after the simple specialization  $e_i \mapsto t$  the function  $SP_n^{(r)}(\mathbf{x})$  reduces to  $A_n^{(r)}(t)$ , the  $r$ -th order Eulerian polynomial, that is the descent generating polynomial of the family of  $r$ -Stirling permutations (defined later). In the case  $r = 1$ ,  $SP_n^{(1)}(\mathbf{x})$  specializes to the classical Eulerian polynomial  $A_n(t) := A_n^{(1)}(t)$ , that is the descent generating polynomial of  $\mathfrak{S}_n$ , and equation (1.1) specializes to the following classical result.

**Theorem 1.6** (Riordan [21]). *We have*

$$\frac{1-t}{1-te^{(1-t)y}} = \sum_{n \geq 0} A_n(t) \frac{y^n}{n!}.$$

In the case  $r = 2$ , we obtain the following analogous result.

**Theorem 1.7.** *We have*

$$\left( \frac{(1-t)y + (1 - e^{y(1-t)})t}{(1-t)^2} \right)^{\langle -1 \rangle} = \sum_{n \geq 1} A_{n-1}^{(2)}(t) \frac{y^n}{n!}.$$

The paper is organized as follows: in Section 2 we discuss Drake's interpretation of compositional inverses of exponential generating functions and use this interpretation to give a version of Theorem 1.5 in terms of the family of normalized labeled binary trees. In Section 3 we use a bijection between normalized labeled binary trees and Stirling permutations to prove Theorem 1.5. We then consider the natural generalization (1.3) of the symmetric functions that appear as coefficients in the right-hand side of equation (1.2) and show in Section 4 that in the base case they are precisely the family of symmetric functions that appear in the right-hand side of equation (1.1). We discuss the interpretation of multiplicative inverses of exponential generating functions and use it to prove Theorem 1.4. In Section 5 we show that under a simple specialization Theorems 1.4 and 1.5 reduce to expressions involving first and second order Eulerian polynomials. Finally, in Section 6 we briefly present other contexts where the symmetric functions  $SP_n^{(1)}(\mathbf{x})$  and  $SP_n^{(2)}(\mathbf{x})$  make an appearance. In particular, these symmetric functions are the generating functions for the Möbius invariants of the maximal intervals of two families of posets. We leave some open questions regarding the cases  $r \geq 3$ .

## 2. BINARY TREES

A tree is a graph that has no loops or cycles. We say that a tree is rooted if one of its nodes is specially marked and called the *root*. For two nodes  $x$  and  $y$  on a rooted tree  $T$ ,  $x$  is said to be the *parent* of  $y$  (and  $y$  the *child* of  $x$ ) if  $x$  is the node that follows  $y$  in the unique path from  $y$  to the root. A node is called a *leaf* if it has no children, otherwise is said to be *internal*. A rooted tree  $T$  is said to be planar if for every internal node  $x$  of  $T$  the set of children of  $x$  is totally ordered. A (*complete planar*) *binary tree* is a planar rooted tree in which every internal node has exactly two children, a left and a right child. For a binary tree  $T$  with  $n$  leaves and  $\sigma \in \mathfrak{S}_n$ , we define a *labeled binary tree*  $(T, \sigma)$  to be the binary tree  $T$  whose  $j$ th leaf from left to right has been labeled  $\sigma(j)$ . We denote by  $\mathcal{BT}_n$  the set of labeled binary trees with  $n$  leaves.

**2.1. Drake's interpretation of the compositional inverse.** In [4] Drake proposes an interesting interpretation of the compositional inverse of an exponential generating function in terms of trees with allowed and forbidden links. This interpretation was also rediscovered by Dotsenko in [3].

Consider rooted trees (either planar or not) that are leaf-labeled with positive integers together with some function, that we call a “valency”, that allows extending the labeling of the leaves to a labeling of the internal nodes. Formally a *valency* is a recursively defined rule that assigns to each node  $x$  (internal or leaf) of a leaf-labeled rooted tree  $T$  a unique element  $v(x)$  such that  $v(x) \in \{v(y) \mid y \text{ a child of } x\}$  if  $x$  is an internal node, or  $v(x) = l$  if  $x$  is a leaf with label  $l$ . The *valency* of a rooted tree  $v(T)$  is defined as the valency of its root.

For two rooted trees  $T_1$  and  $T_2$  with label sets that only coincide in one label that is both a leaf of  $T_1$  and the valency of  $T_2$ , the *composition*  $T_1 \circ T_2$ , is defined to be the tree obtained by deleting the common label from the leaf of  $T_1$  and attaching the root of  $T_2$  instead in its position (see Figure 1). If the above condition for  $T_1$  and  $T_2$  is not satisfied  $T_1 \circ T_2$  is undefined. Note that composition is associative and so expressions like  $T_1 \circ T_2 \circ \cdots \circ T_k$  are well defined.

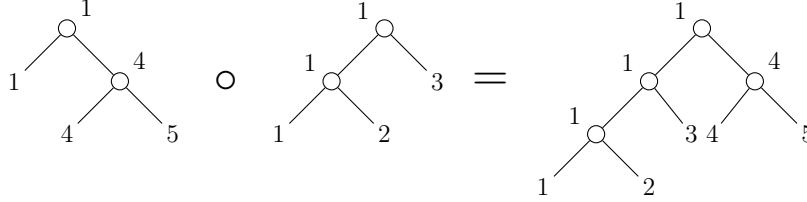


FIGURE 1. Example of composition of leaf-labeled rooted trees. The labels near the internal nodes correspond to the valency of the nodes.

Let  $T_1$  and  $T_2$  be leaf-labeled rooted trees with label sets  $A_1, A_2 \subset \mathbb{P}$  such that  $|A_1| = |A_2|$ .  $T_1$  and  $T_2$  are said to be *equivalent* and write  $T_1 \sim T_2$  if we can obtain  $T_2$  from  $T_1$  by replacing the labels in  $T_1$  according to the unique order preserving bijection between  $A_1$  and  $A_2$ .

If  $\mathcal{A}$  is a set of leaf-labeled trees,  $\mathcal{A}$  is said to have the *label substitution property* if whenever  $T_1 \sim T_2$  then  $T_1 \in \mathcal{A}$  if and only if  $T_2 \in \mathcal{A}$ .  $\mathcal{A}$  is said to have the *unique decomposition property* if for every  $T \in \mathcal{A}$  then  $T \neq T_1 \circ T_2 \circ \cdots \circ T_k$  for trees  $T_j \in \mathcal{A}$  for all  $j$ , i.e.,  $T$  cannot be written as a nontrivial composition of other trees in  $\mathcal{A}$ . A set  $\mathcal{A}$  with these two properties is called an *alphabet* and any tree in  $\mathcal{A}$  is called a *letter*. We can also consider alphabets  $\mathcal{A}_S$  that are formed by colored

letters, that is, pairs  $(T, s)$  where  $T \in \mathcal{A}$  and  $s \in S$  for some set  $S$ . A *link* is the composition of two (colored) letters when defined.

Assume that  $\mathcal{A}_S$  is partitioned into equivalence classes of colored letters and let  $K$  be the set of equivalence classes. For a leaf-labeled tree  $T$  constructed composing letters from  $\mathcal{A}_S$  we denote by  $m_j(T)$  the number of letters of the equivalence class  $j$  that are in  $T$ . We also denote by  $|T| = \sum_{j \in K} m_j(T)$  the total number of letters in  $T$ . Consider now a partition of the set of links into two parts, that we will call from now on *allowed links*  $\mathcal{L}(\mathcal{A}_S)$  and *forbidden links*  $\overline{\mathcal{L}}(\mathcal{A}_S)$ . Let  $\mathcal{T}_S^n$  and  $\overline{\mathcal{T}}_S^n$  for  $n \geq 1$  be the families of trees constructed exclusively with allowed links or exclusively with forbidden links respectively, and whose labels are the elements of the set  $[n]$ , each label occurring exactly once. Define  $\mathcal{T}_S = \cup_{n \geq 1} \mathcal{T}_S^n$  and  $\overline{\mathcal{T}}_S = \cup_{n \geq 1} \overline{\mathcal{T}}_S^n$ . In particular,  $\mathcal{T}_S^1 = \overline{\mathcal{T}}_S^1 = \{\bullet_1\}$  is the tree with a single node labeled 1 and we consider letters in  $\mathcal{A}_S$  as if they are both in  $\mathcal{T}_S$  and  $\overline{\mathcal{T}}_S$ .

Define the monomials

$$(2.1) \quad \mathbf{x}^{m(T)} = \prod_{j \in K} x_j^{m_j(T)},$$

and the generating functions

$$(2.2) \quad \begin{aligned} f_n(\mathbf{x}) &= \sum_{T \in \mathcal{T}_S^n} \mathbf{x}^{m(T)}, \\ \overline{f}_n(\mathbf{x}) &= \sum_{T \in \overline{\mathcal{T}}_S^n} (-1)^{|T|} \mathbf{x}^{m(T)}, \end{aligned}$$

and

$$(2.3) \quad \begin{aligned} F(y) &= \sum_{n \geq 1} f_n(\mathbf{x}) \frac{y^n}{n!}, \\ \overline{F}(y) &= \sum_{n \geq 1} \overline{f}_n(\mathbf{x}) \frac{y^n}{n!}, \end{aligned}$$

where  $y$  and  $x_j$  for  $j \in K$  are indeterminates.

The following theorem of Drake [4] reveals a beautiful algebraic relation between the exponential generating function for the trees constructed using only allowed links and the exponential generating function for the trees constructed using only forbidden links. Its proof is a consequence of the combinatorial interpretation of the composition of exponential generating functions given in [24].

**Theorem 2.1** ([4] Theorem 1.3.3). *We have*

$$(2.4) \quad F^{(-1)}(y) = \overline{F}(y).$$

There is a gap in the argument in the original proof of Theorem 2.1 in [4]. For the sake of completeness we provide a proof of this Theorem fixing this gap.

We begin by considering the set of leaf-labeled rooted trees  $T$  constructed as follows: Starting from a partition  $\pi = (\pi_1, \dots, \pi_\ell)$  of  $[n]$ ,  $T$  is of the form  $T^a \circ T_1^f \circ T_2^f \circ \dots \circ T_\ell^f$  where  $T^a \in \mathcal{T}_S$  and  $T_i^f \in \overline{\mathcal{T}_S}$  for all  $i$ , with the condition that  $T_i^f$  has label set  $\pi_i$  and  $T^a$  has label set  $\{v(T_i^f)\}_{i=1}^\ell$ . Note that different factorizations (different partitions and set of subtrees) of the form above can create the same tree  $T$ . Here we want to consider different factorizations of the same tree  $T$  as different objects. A tree  $T$  together with a factorization as above is called a  $(\mathcal{T}_S, \overline{\mathcal{T}_S})$ -composite tree.

**Lemma 2.2** ([4] Lemma 1.3.2). *The composition  $F(\overline{F}(y))$  is the exponential generating function for  $(\mathcal{T}_S, \overline{\mathcal{T}_S})$ -composite trees  $T$  weighted by  $(-1)^{m_f} \mathbf{x}^{m(T)}$  where  $m_f$  is the number of letters in the forbidden trees.*

*Proof.* This follows from a combinatorial interpretation of composition of exponential generating functions (see [22]).  $\square$

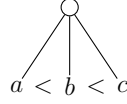
*Proof of Theorem 2.1.* Using Lemma 2.2 we only need to show that the weighted exponential generating function for  $(\mathcal{T}_S, \overline{\mathcal{T}_S})$ -composite trees is equal to  $y$ . We define a sign-reversing involution  $\iota$  on the set of  $(\mathcal{T}_S, \overline{\mathcal{T}_S})$ -composite trees where the only fixed point is the tree with a single node (whose factorization is unique). Let  $\mathfrak{T} = T^a \circ T_1^f \circ T_2^f \circ \dots \circ T_\ell^f$  be a  $(\mathcal{T}_S, \overline{\mathcal{T}_S})$ -composite tree. Now, recursively and starting at the letter  $R$  that contains the root of  $\mathfrak{T}$ , we move to the child of  $R$  with smallest valency among the ones that have an allowed link with  $R$ . The process stops at a letter  $R_0$  that has a forbidden link with every child that is not a leaf. The map works as follows: If  $R_0$  is a letter of  $T^a$  then  $\iota(\mathfrak{T})$  is the factorization where the tree starting at  $R_0$  with all its descendants is part of the forest of trees with forbidden links. If  $R_0$  is not a letter of  $T^a$  then  $\iota(\mathfrak{T})$  is the factorization where  $T^a \circ R_0$  is the new tree with allowed links. Note that the map is well-defined since every child of  $R_0$  is either a leaf or forms a forbidden link with  $R_0$ . Note also that the choice of  $R_0$  does not depend on the factorization of the underlying tree  $T$  and every time we apply the process the role of  $R_0$  changes between being part of the tree with allowed links or being part of a tree in the forest of trees with forbidden links. Then this process is an involution that reverses the weight as defined in Lemma 2.2.  $\square$

*Remark 2.3.* The proof of Theorem 1.3.3 in [4] defines the map  $\iota$  as follows: First select a letter  $R_0$  of  $T^a$  traveling always from the root to



the smallest label that has a letter of  $T^a$  substituted in (it is assumed that the trees are planar and the valency rule chooses the leftmost label in the subtree). For this letter  $R_0$  either all children are leaves or it has at least one child forming a forbidden link with  $R_0$ . If every child is a leaf then  $\iota(\mathfrak{T})$  is the factorization where  $R_0$  is considered as a tree in  $\overline{\mathcal{T}}_S$ . Otherwise let  $R_1$  be the letter substituted into the child with smallest label. If  $R_0 \circ R_1$  is an allowed link then  $\iota(\mathfrak{T})$  is the factorization where  $R_0 \circ R_1$  is part of the tree in  $\mathcal{T}_S$ . Otherwise make  $R_0$  part of the forest of trees with forbidden links.

The issue here is that  $\iota$  is not a well-defined map. For example, assume that we are considering planar letters of the form:



If we decide that the forbidden links are the ones that occur with middle children. Then the  $(\mathcal{T}_S, \overline{\mathcal{T}}_S)$ -composite tree in Figure 2 does not have a well defined image. In the figure, the factorization suggested considers the thick (red) trees together with the single node 1 as the forest of forbidden trees ( $\pi = 1|234|567$ ). It is easy to check that the rule will assign a factorization that does not correspond to a  $(\mathcal{T}_S, \overline{\mathcal{T}}_S)$ -composite tree.

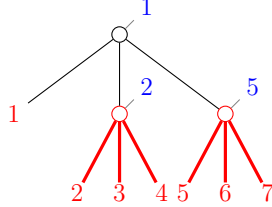
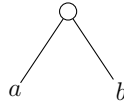
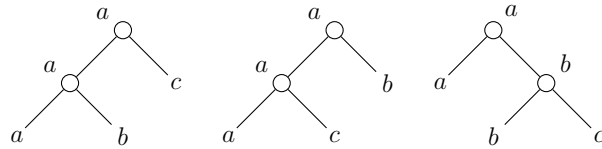


FIGURE 2. Counterexample to the map defined in [4]

In the following we will be using an alphabet of colored planar binary letters of the form:



where  $a < b$  and with colors from  $\mathbb{P}$ . The possible links are of the form:



where  $a < b < c$ .

**2.2. Normalized binary trees.** For each internal node  $x$  of a labeled binary tree, let  $L(x)$  denote the left child of  $x$  and  $R(x)$  denote its right child. For each node  $x$  of a labeled binary tree  $(T, \sigma)$  define its *valency*  $v(x)$  to be the smallest leaf label of the subtree rooted at  $x$ . Figure 3 illustrates the valencies of the internal nodes of a labeled binary tree.

We say that a labeled binary tree is *normalized* if the leftmost leaf of each subtree has the smallest label in the subtree. This is equivalent to requiring that for every internal node  $x$ ,

$$v(x) = v(L(x)).$$

Note that a normalized tree can be regarded as a labeled nonplanar binary tree (or phylogenetic tree) that has been drawn in the plane following the convention above. We denote by  $\text{Nor}_n$  the set of normalized labeled binary trees on label set  $[n]$ . It is well-known that there are  $(2n - 3)!! := 1 \cdot 3 \cdots (2n - 3)$  phylogenetic trees on  $[n]$  and so  $|\text{Nor}_n| = (2n - 3)!!$ .

A *Lyndon tree* is a normalized tree  $(T, \sigma)$  such that for every internal node  $x$  of  $T$  we have

$$(2.5) \quad v(R(L(x))) > v(R(x)).$$

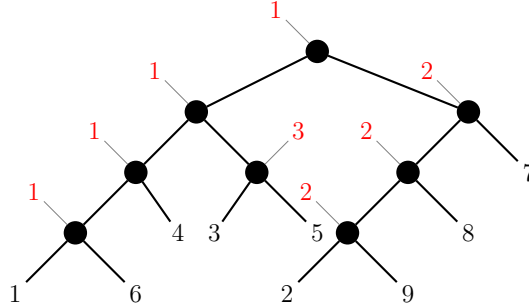


FIGURE 3. Example of a Lyndon tree. The numbers above the lines correspond to the valencies of the internal nodes

We will say that an internal node  $x$  of a labeled binary tree  $(T, \sigma)$  is a *Lyndon node* if (2.5) holds. Hence  $(T, \sigma)$  is a Lyndon tree if and only if it is normalized and all its internal nodes are Lyndon nodes. A Lyndon tree is illustrated in Figure 3. It is known that the set of Lyndon trees with  $n$  leaves gives a basis for the multilinear component of the free Lie algebra on  $n$  generators (see for example [26]).

**2.3. Colored normalized trees.** We will also be considering labeled binary trees with colored internal nodes. A *colored labeled binary tree* is a labeled binary tree such that every internal node  $x$  has been assigned a *color*  $\mathbf{color}(x) \in \mathbb{P}$ . For a weak composition  $\mu \in \text{wcomp}_{n-1}$  we denote  $\mathcal{BT}_\mu$  the set of colored labeled binary trees that contain exactly  $\mu_j$  internal nodes colored  $j$  for each  $j$ .

A *colored Lyndon tree* is a normalized binary tree such that for any node  $x$  that is not a Lyndon node the following condition must be satisfied:

$$(2.6) \quad \mathbf{color}(L(x)) > \mathbf{color}(x).$$

For  $\mu \in \text{wcomp}_{n-1}$ , let  $\mathbf{Lyn}_\mu$  be the set of colored Lyndon trees in  $\mathcal{BT}_\mu$  and  $\mathbf{Lyn}_n = \cup_{\mu \in \text{wcomp}_{n-1}} \mathbf{Lyn}_\mu$ . Note that equation (2.6) implies that the monochromatic Lyndon trees are just the classical Lyndon trees. Figure 4 shows an example of a colored Lyndon tree.

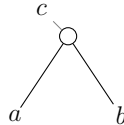
A *colored comb* is a normalized colored binary tree that satisfies the following coloring restriction: for each internal node  $x$  whose right child  $R(x)$  is not a leaf,

$$(2.7) \quad \mathbf{color}(x) > \mathbf{color}(R(x)).$$

Let  $\mathbf{Comb}_\mu$  be the set of colored combs in  $\mathcal{BT}_\mu$  and  $\mathbf{Comb}_n$  the set of all colored combs. Figure 5 shows an example of a colored comb. Note that in a monochromatic comb every right child has to be a leaf and hence they are the classical left combs that are known to give a basis for the multilinear component of the free Lie algebra on  $n$  generators  $\mathcal{Lie}(n)$  (see [26, Proposition 2.3]). The  $\mu$ -colored Lyndon trees and combs generalize the classical Lyndon trees and combs and both give bases for the  $\mathfrak{S}_n$ -module  $\mathcal{Lie}(\mu)$  in [9] (see also [10]).

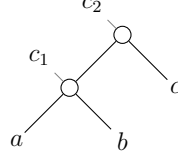
Using Drake's approach we have another perspective to define these types of trees.

Consider the alphabet  $\mathcal{A}_{\mathbb{P}}$  with letters:

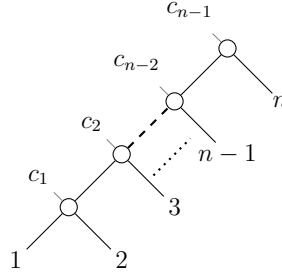


where  $c \in \mathbb{P}$  is any color.

To define the colored Lyndon trees we consider the following forbidden links:



with  $a < b < c$  and  $c_1 \leq c_2$ , i.e., the colors weakly increase towards the root. Then the allowed trees are colored Lyndon trees since they satisfy condition (2.6) and the forbidden trees are of the form:



with  $c_1 \leq c_2 \leq \dots \leq c_{n-1}$ . Since we can completely characterize any such tree by defining how many times the color  $i$  appears among the  $n-1$  nodes for each  $i \in \mathbb{P}$ , using the definition of  $h_n(\mathbf{x})$  in Section 1 we obtain the following expression for the exponential generating series  $\overline{F_{\text{Lyn}}}(y)$  for the forbidden trees.

**Lemma 2.4.** *We have*

$$\overline{F_{\text{Lyn}}}(y) = \sum_{n \geq 1} (-1)^{n-1} h_{n-1}(\mathbf{x}) \frac{y^n}{n!}.$$

**2.3.1. Lyndon type of a normalized tree.** With a normalized tree  $\Upsilon = (T, \sigma) \in \mathbf{Nor}_n$  we can associate a (set) partition  $\pi^{\text{Lyn}}(\Upsilon)$  of the set of internal nodes of  $\Upsilon$ , defined to be the finest partition satisfying the condition:

- for every internal node  $x$  that is not Lyndon,  $x$  and  $L(x)$  belong to the same block of  $\pi^{\text{Lyn}}(\Upsilon)$ .

For the tree in Figure 4, the shaded rectangles indicate the blocks of  $\pi^{\text{Lyn}}(\Upsilon)$ .

Note that the coloring condition (2.6) implies that in a colored Lyndon tree  $\Upsilon$  there are no repeated colors in each block  $B$  of the partition  $\pi^{\text{Lyn}}(\Upsilon)$  associated with  $\Upsilon$ . Hence after choosing a set of  $|B|$  colors for the internal nodes in  $B$  there is a unique way to assign the different colors such that the colored tree  $\Upsilon$  is a colored Lyndon tree (the colors must decrease towards the root in each block of  $\pi^{\text{Lyn}}(\Upsilon)$ ).

Define the *Lyndon type*  $\lambda^{\text{Lyn}}(\Upsilon)$  of a normalized tree (colored or uncolored)  $\Upsilon$  to be the (integer) partition whose parts are the block sizes of  $\pi^{\text{Lyn}}(\Upsilon)$ . For the tree  $\Upsilon$  in Figure 4, we have  $\lambda^{\text{Lyn}}(\Upsilon) = (3, 2, 2, 1)$ .

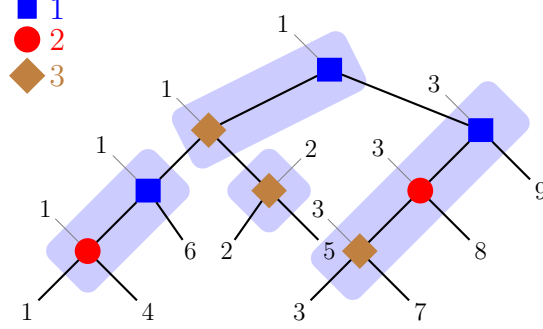


FIGURE 4. Example of a colored Lyndon tree of type  $(3, 2, 2, 1)$ . The numbers above the lines correspond to the valencies of the internal nodes

**Proposition 2.5.** *We have*

$$F_{\text{Lyn}}(y) = \sum_{n \geq 1} \sum_{\Upsilon \in \text{Nor}_n} e_{\lambda^{\text{Lyn}}(\Upsilon)}(\mathbf{x}) \frac{y^n}{n!}.$$

*Proof.* For a colored labeled binary tree  $\Psi$  we define its *content*  $\mu(\Psi)$  to be the weak composition  $\mu$  where  $\mu(i)$  is the number of internal nodes of  $\Psi$  that have color  $i$ . Let  $\tilde{\Psi}$  denote the underlying uncolored labeled binary tree of  $\Psi$ . Note that the comments above imply that for  $\Upsilon \in \text{Nor}_n$ , the generating function of colored Lyndon trees associated with  $\Upsilon$  is

$$(2.8) \quad \sum_{\substack{\Psi \in \text{Lyn}_n \\ \tilde{\Psi} = \Upsilon}} \mathbf{x}^{\mu(\Psi)} = e_{\lambda^{\text{Lyn}}(\Upsilon)}(\mathbf{x}).$$

Indeed, the internal nodes in a block of size  $i$  in the partition  $\pi^{\text{Lyn}}(\Upsilon)$  can be colored uniquely with any set of  $i$  different colors and so the contribution from this block of  $\pi^{\text{Lyn}}(\Upsilon)$  to the generating function in (2.8) is  $e_i(\mathbf{x})$ . Then

$$\begin{aligned} \sum_{\Psi \in \text{Lyn}_n} \mathbf{x}^{\mu(\Psi)} &= \sum_{\Upsilon \in \text{Nor}_n} \sum_{\substack{\Psi \in \text{Lyn}_n \\ \tilde{\Psi} = \Upsilon}} \mathbf{x}^{\mu(\Psi)} \\ &= \sum_{\Upsilon \in \text{Nor}_n} e_{\lambda^{\text{Lyn}}(\Upsilon)}(\mathbf{x}), \end{aligned}$$

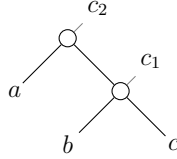
with the last equality following from (2.8).  $\square$

We obtain the following theorem as a corollary of Theorem 2.1.

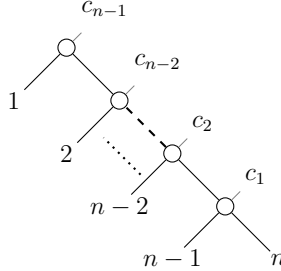
**Theorem 2.6** ([9, Theorems 1.5 and 4.3]). *We have*

$$(2.9) \quad \left( \sum_{n \geq 1} (-1)^{n-1} h_{n-1}(\mathbf{x}) \frac{y^n}{n!} \right)^{\langle -1 \rangle} = \sum_{n \geq 1} \left( \sum_{\Upsilon \in \text{Nor}_n} e_{\lambda^{\text{Lyn}}(\Upsilon)}(\mathbf{x}) \right) \frac{y^n}{n!}.$$

To define the colored combs now we consider forbidden links of the form:



with  $a < b < c$  and  $c_1 \geq c_2$ , i.e., the colors weakly increase towards the root. Then the allowed trees are colored combs and the forbidden trees look like



with  $c_1 \geq c_2 \geq \dots \geq c_{n-1}$ . Then following the same argument as the one before Lemma 2.4 we obtain the following expression for the exponential generating series  $\overline{F_{\text{Comb}}}(y)$  of the forbidden trees.

**Lemma 2.7.** *We have*

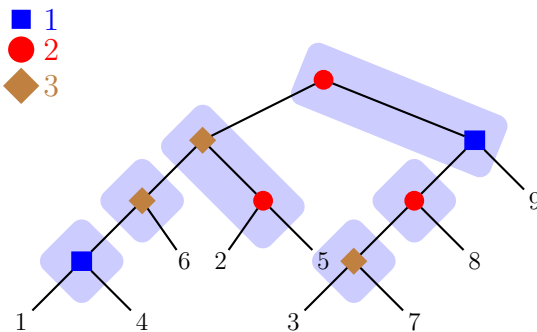
$$\overline{F_{\text{Comb}}}(y) = \sum_{n \geq 1} (-1)^{n-1} h_{n-1}(\mathbf{x}) \frac{y^n}{n!}.$$

**2.3.2. Comb type of a normalized tree.** We can associate a new type to each  $\Upsilon \in \text{Nor}_n$  in the following way: Let  $\pi^{\text{Comb}}(\Upsilon)$  be the finest (set) partition of the set of internal nodes of  $\Upsilon$  satisfying

- for every pair of internal nodes  $x$  and  $y$  such that  $y$  is a right child of  $x$ ,  $x$  and  $y$  belong to the same block of  $\pi^{\text{Comb}}(\Upsilon)$ .

We define the *comb type*  $\lambda^{\text{Comb}}(\Upsilon)$  of  $\Upsilon$  to be the (integer) partition whose parts are the sizes of the blocks of  $\pi^{\text{Comb}}(\Upsilon)$ .

In the same manner than for Proposition 2.5 and Theorem 2.6 we derive the corresponding results for colored combs.

$$F_{Comb}(y) = \sum_{n \geq 1} \sum_{\Upsilon \in \text{Nor}_n} e_{\lambda_{Comb}(\Upsilon)}(\mathbf{x}) \frac{y^n}{n!}.$$
$$\left( \sum_{n \geq 1} (-1)^{n-1} h_{n-1}(\mathbf{x}) \frac{y^n}{n!} \right)^{\langle -1 \rangle} = \sum_{n \geq 1} \left( \sum_{\Upsilon \in \text{Nor}_n} e_{\lambda^{\text{Comb}}(\Upsilon)}(\mathbf{x}) \right) \frac{y^n}{n!}.$$


*Remark 2.10.* In [9] Theorem 2.6 is proved using a different technique. The proof involves the recursive definition of the Möbius invariant and an EL-labeling of a poset of partitions weighted by weak compositions where the ascent-free (or falling) chains coming from the EL-labeling are naturally described by colored Lyndon trees. The proof of Theorem 2.9 in [9] is a corollary of Theorems 2.6 and 2.11.

Lemmas 2.4 and 2.7 and Theorem 2.1 give a new proof of the following theorem proved bijectively in [9].

**Theorem 2.11.** [9, Theorem 5.4] *For every  $\mu \in \text{wcomp}$ ,*

$$|\text{Lyn}_\mu| = |\text{Comb}_\mu|.$$

*Remark 2.12.* It is interesting to note that under Drake's interpretation Theorem 2.11 becomes somewhat more transparent: The two sets  $\text{Lyn}_n$  and  $\text{Comb}_n$  are constructed using the same alphabet avoiding two different sets of forbidden links that are in bijection with each other. It is also interesting that both types of forbidden trees have a kind of "shape duality", colored Lyndon trees have forbidden trees that look like "left-combs" and colored combs have forbidden trees that look like "right-combs".

### 3. STIRLING PERMUTATIONS

Now we consider permutations of the multiset  $\{1, 1, 2, 2, \dots, n, n\}$  satisfying the condition that all the numbers between the two occurrences of any fixed number  $m$  are larger than  $m$ , that is, multiset permutations  $\theta$  satisfying the following condition:

$$(3.1) \quad \text{if } i < j < k \text{ and } \theta_i = \theta_k = m \text{ then } \theta_j \geq m.$$

To this family, denoted  $\mathcal{Q}_n$  for  $n \geq 0$ , belongs for example the permutation 12234431 but not the permutation 11322344 since 2 is less than 3 and 2 is between the two occurrences of 3. The permutations in  $\mathcal{Q}_n$  were introduced by Gessel and Stanley in [7] and are known as *Stirling Permutations*.

For  $\theta \in \mathcal{Q}_n$ , assuming always that  $\theta_0 = \theta_{2n+1} = 0$ , consider the sets

$$(3.2) \quad \begin{aligned} \text{DES}(\theta) &= \{i \mid \theta_i > \theta_{i+1}\}, \\ \text{ASC}(\theta) &= \{i \mid \theta_i < \theta_{i+1}\} \text{ and} \\ \text{PLA}(\theta) &= \{i \mid \theta_i = \theta_{i+1}\}. \end{aligned}$$

These are respectively the sets of *descents*, *ascents* and *plateaux* defined in [7] and [1]. Let  $\text{des}(\theta) = |\text{DES}(\theta)|$ ,  $\text{asc}(\theta) = |\text{ASC}(\theta)|$  and  $\text{pla}(\theta) = |\text{PLA}(\theta)|$  be their cardinalities. It is an immediate observation that the statistics  $\text{des}$  and  $\text{asc}$  are equidistributed. For this it is enough to consider the function  $\rho : \mathcal{Q}_n \rightarrow \mathcal{Q}_n$  that reverses a permutation  $\rho(\theta)_i = \theta_{2n+1-i}$ . Bóna ([1]) proved that these 2 statistics are also equidistributed with  $\text{pla}$  by showing that they satisfy the same recurrence relation. Janson, Kuba and Panholzer [14] gave a simple combinatorial proof using a bijection of Gessel between Stirling permutations and increasing ternary trees.

The numbers coming out of any of these three statistics are known as the *second-order Eulerian numbers* (see [11]). This terminology emphasizes that the Stirling permutations are the second case ( $r =$



2) of a more general family  $\mathcal{Q}_n(r)$  of permutations of the multiset  $\{1^r, 2^r, \dots, n^r\}$  satisfying condition (3.1). Note that  $\mathcal{Q}(1) = \mathfrak{S}_n$  and  $\mathcal{Q}_n(2) = \mathcal{Q}_n$ . These more general multiset permutations have been also studied (see [20, 19, 18, 14, 16] and [12]) with the name of *r-Stirling permutations* or *r-multipermutations*. We borrow the terminology in [11] and call in general any statistic that is equidistributed with the descent statistic in  $\mathcal{Q}_n(r)$  an *rth-order Eulerian statistic*.

From a permutation in  $\mathcal{Q}_{n-1}$  we can obtain a permutation in  $\mathcal{Q}_n$  by inserting the consecutive labels  $nn$  in  $2n - 1$  possible positions. Inductively and starting in  $\mathcal{Q}_1 = \{11\}$  this implies that  $|\mathcal{Q}_n| = 1 \cdot 3 \cdots (2n - 1) = (2n - 1)!!$ .

**3.1. Type of a Stirling permutation.** In this section we define several types associated to a Stirling permutation. These types were introduced in [9].

A *segment*  $u$  of a Stirling permutation  $\theta = \theta_1\theta_2 \cdots \theta_{2n}$  is a subword of  $\theta$  of the form  $u = \theta_i\theta_{i+1} \cdots \theta_{i+\ell}$ , i.e., all the letters of  $u$  are adjacent in  $\theta$ . A *block* in a Stirling permutation  $\theta$  is a segment of  $\theta$  that starts and ends with the same letter. For example, 455774 is a block of 12245577413366. We define  $B_\theta(a)$  to be the block of  $\theta$  that starts and ends with the letter  $a$ , and define  $\dot{B}_\theta(a)$  to be the segment obtained from  $B_\theta(a)$  after removing the two occurrences of the letter  $a$ . For example,  $B_\theta(1) = 1224557741$  in  $\theta = 12245577413366$  and  $\dot{B}_\theta(1) = 22455774$ .

We call  $(a, b)$  an *ascending adjacent pair* of  $\theta \in \mathcal{Q}_n$  if  $a < b$  and the blocks  $B_\theta(a)$  and  $B_\theta(b)$  are adjacent in  $\theta$ , i.e.,  $\theta = \theta' B_\theta(a) B_\theta(b) \theta''$ . An *ascending adjacent sequence* of  $\theta$  of length  $k$  is a subsequence  $a_1 < a_2 < \cdots < a_k$  such that  $(a_j, a_{j+1})$  is an ascending adjacent pair for  $j = 1, \dots, k-1$ . Similarly, we call  $(a, b)$  a *terminally nested pair* if  $a < b$  and the block  $B_\theta(b)$  is the last block in  $\dot{B}_\theta(a)$ , i.e.,  $\dot{B}_\theta(a) = \theta' B_\theta(b)$  for some Stirling permutation  $\theta'$ . A *terminally nested sequence* of  $\theta$  of length  $k$  is a subsequence  $a_1 < a_2 < \cdots < a_k$  such that  $(a_j, a_{j+1})$  is a terminally nested pair for  $j = 1, \dots, k-1$ . If we apply the map  $\rho$  that reverses the permutation to the two definitions above we obtain the notions of *descending adjacent* and *initially nested* pairs and sequences.

We can associate a *type* to a Stirling permutation  $\theta \in \mathcal{Q}_n$  using the different types of sequences defined above. We define the *ascending adjacent type*  $\lambda^{\text{AA}}(\theta)$ , to be the partition whose parts are the lengths of maximal ascending adjacent sequences; the *terminally nested type*  $\lambda^{\text{TN}}(\theta)$ , to be the partition whose parts are the lengths of maximal terminally nested sequences; the *descending adjacent type*  $\lambda^{\text{DA}}(\theta)$ , to be

the partition whose parts are the lengths of maximal descending adjacent sequences; and the *initially nested type*  $\lambda^{\text{IN}}(\theta)$ , to be the partition whose parts are the lengths of maximal initially nested sequences.

**Example 3.1.** If  $\theta = 158851244667729933$ , the maximal ascending adjacent sequences are 129, 467, 3, 5 and 8 and  $\lambda^{\text{AA}}(\theta) = (3, 3, 1, 1, 1)$ . The maximal descending adjacent sequences are 1, 2, 93, 4, 6, 7, 5 and 8 and  $\lambda^{\text{DA}}(\theta) = (2, 1, 1, 1, 1, 1, 1)$ . The maximal terminally nested sequences are 158, 27, 3, 4, 6 and 9 and  $\lambda^{\text{TN}}(\theta) = (3, 2, 1, 1, 1, 1)$ . The maximal initially nested sequences are 158, 24, 6, 7, 9 and 3 and  $\lambda^{\text{IN}}(\theta) = (3, 2, 1, 1, 1, 1)$ . All of them are partitions of  $n = 9$ .

Note that since every descent in  $\theta$  occurs at the end of a maximal ascending adjacent sequence then  $\ell(\lambda^{\text{AA}}(\theta)) = \text{des}(\theta)$  where  $\ell(\lambda)$  indicates the number of parts of a partition  $\lambda$ . So  $\lambda^{\text{AA}}$  is a refinement of the des statistic. In the same manner since each plateau can be considered as occurring either at the end of a maximal terminally nested sequence or at the end of a maximal initially nested sequence then  $\ell(\lambda^{\text{TN}}(\theta)) = \ell(\lambda^{\text{IN}}(\theta)) = \text{pla}(\theta)$ . The proper happens for  $\ell(\lambda^{\text{DA}}(\theta)) = \text{asc}(\theta)$ . Table 1 gives the values of des, asc, pla,  $\lambda^{\text{AA}}$ ,  $\lambda^{\text{DA}}$ ,  $\lambda^{\text{TN}}$  and  $\lambda^{\text{IN}}$  for  $n = 3$ . For  $n = 3$  it happens that  $\lambda^{\text{TN}}$  and  $\lambda^{\text{IN}}$  are equal but this is not true in general.

$\theta$	$\text{des}(\theta)$	$\text{asc}(\theta)$	$\text{pla}(\theta)$	$\lambda^{\text{AA}}(\theta)$	$\lambda^{\text{DA}}(\theta)$	$\lambda^{\text{TN}}(\theta)$	$\lambda^{\text{IN}}(\theta)$
112233	1	3	3	(3)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)
113322	2	2	3	(2, 1)	(2, 1)	(1, 1, 1)	(1, 1, 1)
221133	2	2	3	(2, 1)	(2, 1)	(1, 1, 1)	(1, 1, 1)
223311	2	2	3	(2, 1)	(2, 1)	(1, 1, 1)	(1, 1, 1)
331122	2	2	3	(2, 1)	(2, 1)	(1, 1, 1)	(1, 1, 1)
122331	2	3	2	(2, 1)	(1, 1, 1)	(2, 1)	(2, 1)
112332	2	3	2	(2, 1)	(1, 1, 1)	(2, 1)	(2, 1)
133122	2	3	2	(2, 1)	(1, 1, 1)	(2, 1)	(2, 1)
122133	2	3	2	(2, 1)	(1, 1, 1)	(2, 1)	(2, 1)
133221	3	2	2	(1, 1, 1)	(2, 1)	(2, 1)	(2, 1)
221331	3	2	2	(1, 1, 1)	(2, 1)	(2, 1)	(2, 1)
233211	3	2	2	(1, 1, 1)	(2, 1)	(2, 1)	(2, 1)
331221	3	2	2	(1, 1, 1)	(2, 1)	(2, 1)	(2, 1)
123321	3	3	1	(1, 1, 1)	(1, 1, 1)	(3)	(3)
332211	3	1	3	(1, 1, 1)	(3)	(1, 1, 1)	(1, 1, 1)

TABLE 1. Stirling permutations and statistics for  $n = 3$ .

The following proposition was proved in [9].

**Proposition 3.2** ([9, Proposition 4.6]). *There is a bijection  $\xi : \mathcal{Q}_n \rightarrow \mathcal{Q}_n$  that satisfies:*

- (1)  *$(i, j)$  is an ascending adjacent pair in  $\theta$  if and only if  $(i, j)$  is a terminally nested pair in  $\xi(\theta)$ ,*
- (2)  *$\lambda^{TN}(\xi(\theta)) = \lambda^{AA}(\theta)$ .*

From Proposition 3.2 we see that  $\lambda^{AA}$  and  $\lambda^{TN}$  are equidistributed on  $\mathcal{Q}_n$ . Since the map  $\rho$  that reverses  $\theta$  (defined above) also implies  $\lambda^{AA} \cong \lambda^{DA}$  and  $\lambda^{TN} \cong \lambda^{IN}$  we have as corollaries the following two results.

**Theorem 3.3.** *The types  $\lambda^{AA}$ ,  $\lambda^{DA}$ ,  $\lambda^{TN}$  and  $\lambda^{IN}$  are equidistributed.*

**Corollary 3.4** ([1]). *The statistics  $\text{des}$ ,  $\text{asc}$  and  $\text{pla}$  are equidistributed.*

*Remark 3.5.* Theorem 3.3 can be proved in a different way using a bijection of Gessel between Stirling permutations and increasing planar ternary trees (see [14]). The idea is that from the perspective of increasing planar ternary trees (reading the types after the bijection), the equidistributivity of the types  $\lambda^{AA}$ ,  $\lambda^{DA}$ ,  $\lambda^{TN}$  and  $\lambda^{IN}$  is a consequence of the bijection in the set of increasing planar ternary trees defined by reordering simultaneously the 3 children of every internal node of a ternary tree using a fixed permutation. Since there are 6 permutations in  $\mathfrak{S}_3$ , proving Theorem 3.3 in this way also reveals that there are two other different types equidistributed with the ones discussed here. We leave the details of this proof to the reader.

The link between normalized trees and Stirling permutations is given by the following proposition in [9].

**Proposition 3.6** ([9, Proposition 4.8]). *There is a bijection<sup>1</sup>  $\gamma : \text{Nor}_n \rightarrow \mathcal{Q}_{n-1}$  that satisfies for each  $\Upsilon \in \text{Nor}_n$*

- (1)  *$\lambda^{AA}(\gamma(\Upsilon)) = \lambda^{Ly^n}(\Upsilon)$ ,*
- (2)  *$\lambda^{TN}(\gamma(\Upsilon)) = \lambda^{Comb}(\Upsilon)$ .*

Combining the results in Propositions 3.2 and 3.6; and Theorems 2.6 and 2.9 we obtain Theorem 1.5.

**Theorem 3.7** (Theorem 1.5). *We have*

$$(3.3) \quad \left( \sum_{n \geq 1} (-1)^{n-1} h_{n-1}(\mathbf{x}) \frac{y^n}{n!} \right)^{\langle -1 \rangle} = \sum_{n \geq 1} \left( \sum_{\theta \in \mathcal{Q}_{n-1}} e_{\lambda(\theta)}(\mathbf{x}) \right) \frac{y^n}{n!},$$

where  $\lambda(\theta)$  is one of the types  $\lambda^{AA}$ ,  $\lambda^{DA}$ ,  $\lambda^{TN}$  and  $\lambda^{IN}$  for  $\theta \in \mathcal{Q}_n$ .

---

<sup>1</sup>This bijection appeared first in [3]

**3.2.  $r$ -Stirling permutations or  $r$ -multipermutations.** An  $r$ -*Stirling permutation* or  $r$ -*multipermutation* is a permutation of the multiset  $\{1^r, 2^r, \dots, n^r\}$  satisfying condition (3.1). We denote the set of  $r$ -Stirling permutations by  $\mathcal{Q}_n(r)$ . For example, 123332215551466644 is in  $\mathcal{Q}_n(3)$ . In particular  $\mathcal{Q}_n(2) = \mathcal{Q}_n$  and  $\mathcal{Q}_n(1) = \mathfrak{S}_n$ . We want to extend the right-hand side of equation (3.3) to the generality of  $r$ -Stirling permutations.

For a permutation  $\sigma \in \mathcal{Q}_n(r)$ , assuming always that  $\sigma_0 = \sigma_{rn+1} = 0$ , we define the sets DES and ASC and the statistics des and asc as before. Let  $n(\sigma, i)$  denote the number of occurrences of the label  $\sigma_i$  in the subword  $\sigma_1\sigma_2\cdots\sigma_i$ . We use a refinement of PLA defined in [14], the set

$$\text{PLA}_j(\sigma) = \{i \mid \sigma_i = \sigma_{i+1}, n(\sigma, i) = j\}$$

of  $j$ -plateaux (plateaux between the occurrences  $j$  and  $j+1$  of a label) and the statistic  $\text{pla}_j(\sigma) = |\text{PLA}_j(\sigma)|$  its cardinality. In [14] is shown that des, asc and  $\text{pla}_j$  are equidistributed in  $\mathcal{Q}_n(r)$ .

**3.2.1. Types on the set of  $r$ -Stirling permutations.** A block  $B_\theta(a)$  in an  $r$ -Stirling permutation  $\theta$  is a segment of  $\theta$  that starts and ends with  $a$  and contains all the occurrences of  $a$  in  $\theta$ . For example,  $B_\theta(1) = 122214555441$  is a block of  $\theta = 122214555441333$ . Removing all occurrences of  $a$  in  $B_\theta(a)$  gives a sequence  $\mathring{B}_\theta(a)$  of (possibly empty)  $r$ -Stirling permutations  $\mathring{B}_\theta(a)_j$  for  $j = 1, \dots, r-1$ . For example  $\mathring{B}_\theta(1) = (222, 455544)$ .

We call  $(a, b)$  an *ascending adjacent pair* in  $\theta \in \mathcal{Q}_n(r)$  if  $a < b$  and the blocks  $B_\theta(a)$  and  $B_\theta(b)$  are adjacent in  $\theta$ , i.e.,  $\theta = \theta' B_\theta(a) B_\theta(b) \theta''$ . An *ascending adjacent sequence* of  $\theta$  of length  $k$  is a subsequence  $a_1 < a_2 < \dots < a_k$  such that  $(a_j, a_{j+1})$  is an ascending adjacent pair for  $j = 1, \dots, k-1$ . Similarly, for  $j \in [r-1]$  we call  $(a, b)$  a  *$j$ -terminally nested pair* if  $a < b$  and the block  $B_\theta(b)$  is the last block in  $\mathring{B}_\theta(a)_j$ , i.e.,  $\mathring{B}_\theta(a)_j = \theta' B_\theta(b)$  for some  $r$ -Stirling permutation  $\theta'$ . A  *$j$ -terminally nested sequence* of  $\theta$  of length  $k$  is a subsequence  $a_1 < a_2 < \dots < a_k$  such that  $(a_s, a_{s+1})$  is a  $j$ -terminally nested pair for  $s = 1, \dots, k-1$ . If we apply the map  $\rho$  that reverses the permutation to the two definitions above we obtain the notions of *descending adjacent* and  *$j$ -initially nested* pairs and sequences.

We then associate a *type* to an  $r$ -Stirling permutation  $\theta \in \mathcal{Q}_n(r)$  in different ways according to the lengths of maximal sequences of a given type as before. We define in this way the *ascending adjacent type*  $\lambda^{\text{AA}}(\theta)$ , the  *$j$ -terminally nested type*  $\lambda_j^{\text{TN}}(\theta)$ , the *descending adjacent type*  $\lambda^{\text{DA}}(\theta)$  and the  *$j$ -initially nested type*  $\lambda_j^{\text{IN}}(\theta)$ . Note that similar to

the case  $r = 2$ , in the general case  $\lambda^{AA}$  refines  $\text{des}$ ,  $\lambda^{DA}$  refines  $\text{asc}$  and both  $\lambda_j^{TN}$  and  $\lambda_j^{IN}$  refine  $\text{pla}_j$ .

The proof of the following theorem is similar to the proof of Theorem 3.3 in [9].

**Theorem 3.8.** *The types  $\lambda^{AA}$ ,  $\lambda^{DA}$ ,  $\lambda_j^{TN}$  and  $\lambda_j^{IN}$  for all  $j = 1, \dots, r-1$  are equidistributed.*

*Remark 3.9.* We can also prove Theorem 3.8 following the same idea discussed in Remark 3.5. This time we use instead a more general bijection of Gessel between  $r$ -Stirling permutations and increasing planar  $(r+1)$ -ary trees.

With the more general definitions in place we can consider the family of symmetric functions

$$(3.4) \quad SP_n^{(r)}(\mathbf{x}) = \sum_{\theta \in \mathcal{Q}_n(r)} e_{\lambda(\theta)}(\mathbf{x}),$$

where  $\lambda(\theta)$  is any of the types of  $\theta$  defined above.

The question is to determine if this more general definition provide interesting results and has any combinatorial applications for some  $r \neq 2$ . We will show in Section 4 that for  $r = 1$  there is a positive answer with a very similar story to the one for  $r = 2$ .

#### 4. STANDARD PERMUTATIONS=1-STIRLING PERMUTATIONS

In this section we prove Theorem 1.4. To prove this theorem we first introduce a combinatorial interpretation of the multiplicative inverse of an exponential generating function in terms of words with allowed and forbidden links that follows from a theorem discovered by Fröberg [6], Carlitz-Scoville-Vaughan [2] and Gessel [8]. The theory outlined in [2] and [8] is more general and apply to a larger family of counting algebras (as defined in [8]) and not only to exponential generating functions, here we give a simplified description that applies to the exponential generating function result.

##### 4.1. Combinatorial interpretation of the multiplicative inverse.

We call a *word* a permutation of some finite label set  $A \subset \mathbb{P}$ , i.e., a string of labels in  $\mathbb{P}$  that contains no repeated letters. For example, 1243 and 2147 are both examples of words but 122 is not since the label 2 is repeated. In particular any label by itself and the empty word  $\emptyset$  are considered words. For any two words  $w_1$  and  $w_2$  that have disjoint label sets  $A_1, A_2 \subset \mathbb{P}$  we define the *product*  $w_1 w_2$  to be the word with label set  $A_1 \cup A_2$  constructed by concatenation. If these conditions are not satisfied the product is not defined. For example if

$w_1 = 1345$  and  $w_2 = 276$  then  $w_1 w_2 = 1345276$ . Note that the product is associative and so expressions like  $w_1 w_2 \cdots w_k$  are well defined. Two words  $w_1$  and  $w_2$  with label sets  $A_1, A_2 \subset \mathbb{P}$  such that  $|A_1| = |A_2|$  are said to be *equivalent*, and we write  $w_1 \sim w_2$ , if we can obtain  $w_2$  from  $w_1$  by replacing the labels in  $w_1$  according to the unique order preserving bijection between  $A_1$  and  $A_2$ . If  $\mathcal{A}$  is a set of words,  $\mathcal{A}$  is said to have the *label substitution property* if whenever  $w_1 \sim w_2$  then  $w_1 \in \mathcal{A}$  if and only if  $w_2 \in \mathcal{A}$ . In a set  $\mathcal{A}$  of words, we call a word  $w \in \mathcal{A}$  *irreducible* if  $w$  is a nonempty word such that  $w = w_1 w_2$  and  $w_1, w_2 \in \mathcal{A}$  imply either  $w_1 = w$  or  $w_2 = w$ , i.e., a word that cannot be decomposed as the concatenation of other words in  $\mathcal{A}$ . A set  $\mathcal{A}$  of words is said to have the *unique decomposition property* if all its words are irreducible. A set  $\mathcal{A}$  of words that has the label substitution property and the unique decomposition property is called an *alphabet* and its elements are called the *letters* of the alphabet. The set of words (including the empty word) that can be constructed as the product of letters in  $\mathcal{A}$  is denoted  $\mathcal{A}^*$ . We can also consider alphabets  $\mathcal{A}_S$  that are formed by colored letters, i.e., pairs  $(a, s)$  where  $a \in \mathcal{A}$  and  $s \in S$  for some set  $S$ . We call a *link* the product of two (colored) letters. The set of links is then  $\mathcal{A}_S \times \mathcal{A}_S$ .

Assume that  $K$  is the set of equivalence classes of colored letters in  $\mathcal{A}_S$ . For a word  $w \in \mathcal{A}_S^*$  we denote by  $m_j(w)$  the number of letters of color  $j$  that are present in  $w$  and  $|w| := \sum_{j \in K} m_j(w)$  the *length* of  $w$ . Consider a partition of the set  $\mathcal{A}_S \times \mathcal{A}_S$  of links into two parts that we call the *allowed links*  $\mathcal{L}(\mathcal{A}_S)$  and *forbidden links*  $\overline{\mathcal{L}(\mathcal{A}_S)}$ . Let  $\mathcal{W}_S^n$  be the set of words with underlying label set  $[n]$  for  $n \geq 0$  constructed with only allowed links and let  $\overline{\mathcal{W}_S^n}$  the ones constructed using only forbidden links. Define  $\mathcal{W}_S = \cup_{n \geq 0} \mathcal{W}_S^n$  and  $\overline{\mathcal{W}_S} = \cup_{n \geq 0} \overline{\mathcal{W}_S^n}$ . In particular,  $\mathcal{W}_S^0 = \overline{\mathcal{W}_S^0} = \emptyset$  and we consider letters in  $\mathcal{A}_S$  as if they are both in  $\mathcal{W}_S$  and  $\overline{\mathcal{W}_S}$ .

Define the monomials

$$X^{m(w)} = \prod_{j \in K} x_j^{m_j(w)},$$

and the generating functions

$$\begin{aligned} f_n(X) &= \sum_{w \in \mathcal{W}_S^n} X^{m(w)}, \\ \overline{f}_n(X) &= \sum_{w \in \overline{\mathcal{W}_S^n}} (-1)^{|w|} X^{m(w)}, \end{aligned}$$

and

$$F(y) = \sum_{n \geq 0} f_n(X) \frac{y^n}{n!},$$

$$\overline{F}(y) = \sum_{n \geq 0} \overline{f}_n(X) \frac{y^n}{n!}.$$

**Theorem 4.1** (c.f.[8]). *We have*

$$F^{-1}(y) = \overline{F}(y).$$

For the sake of completeness we provide a proof of Theorem 4.1. The idea of the proof is the one that appears in [8] where a more general version of this theorem is proven.

To prove Theorem 4.1 we consider the set of words of the form  $w = w_1 w_2$  where  $w_1 \in \mathcal{W}_S$  and  $w_2 \in \overline{\mathcal{W}_S}$ . Note that if  $w \neq \emptyset$  then  $w$  has exactly two different factorizations. Indeed, if  $w = a_1 a_2 \dots a_n$  then either there is a number  $1 \leq k < n$  such that  $a_k a_{k+1} \in \mathcal{L}(\mathcal{A}_S)$  but  $a_{k+1} a_{k+2} \in \overline{\mathcal{L}(\mathcal{A}_S)}$ , or  $a_k a_{k+1} \in \mathcal{L}(\mathcal{A}_S)$  for all  $k$  (in which case we let  $k = n$ ) or  $a_k a_{k+1} \in \overline{\mathcal{L}(\mathcal{A}_S)}$  for all  $k$  (in which case we let  $k = 0$ ). Assuming that  $a_0 = a_{n+1} = \emptyset$  the two valid factorizations of  $w$  are  $(a_0 \dots a_{k+1})(a_{k+2} \dots a_{n+1})$  and  $(a_0 \dots a_k)(a_{k+1} \dots a_{n+1})$ . Here we want to consider the two different factorizations of  $w$  as different objects. We call these objects (a word  $w$  together with its factorization)  $(\mathcal{W}_S, \overline{\mathcal{W}_S})$ -composite words.

**Lemma 4.2.** *The multiplication  $F(y)\overline{F}(y)$  is the exponential generating function for  $(\mathcal{W}_S, \overline{\mathcal{W}_S})$ -composite words  $w$  weighted by  $(-1)^{m_f} \mathbf{x}^{m(w)}$  where  $m_f$  is the number of letters in the forbidden word.*

*Proof.* This follows from the combinatorial interpretation of multiplication of exponential generating functions in [22].  $\square$

*Proof of Theorem 4.1.* Using Lemma 4.2 we only need to show that the weighted exponential generating function for  $(\mathcal{W}_S, \overline{\mathcal{W}_S})$ -composite words is equal to 1. We define a sign-reversing involution  $\iota$  on the set of  $(\mathcal{W}_S, \overline{\mathcal{W}_S})$ -composite words where the only fixed point is the empty word (whose factorization is unique). Let  $w$  be a nonempty  $(\mathcal{W}_S, \overline{\mathcal{W}_S})$ -composite word, by the comments above we know that the underlying permutation of  $w$  can be associated with two different factorizations  $w$  and  $w'$ . We then define  $\iota(w) = w'$  if  $w \neq \emptyset$  and  $\iota(\emptyset) = \emptyset$ . This process is an involution that reverses the weight as defined in Lemma 4.2 since  $w$  and  $w'$  differ by one in the number of letters in the forbidden word.  $\square$

**4.2. Colored permutations.** Let  $S$  be any subset of  $\mathbb{P}$ . A *colored permutation* is a permutation  $\sigma \in \mathfrak{S}_n$  in which each letter  $j \in [n]$  has been assigned a color  $\mathbf{color}(j) \in S$  with the condition that for every occurrence of an ascending adjacent pair  $\sigma(j) < \sigma(j+1)$  in  $\sigma$  it must happen that  $\mathbf{color}(\sigma(j)) > \mathbf{color}(\sigma(j+1))$ . For example for  $S = [3]$ ,  $\sigma = 1^3 2^2 4^1 3^3 5^2$  is a colored permutation with the colored letters  $(i, \mathbf{color}(i))$  represented as  $i^{\mathbf{color}(i)}$ . Since in any ascending adjacent sequence of  $\sigma \in \mathfrak{S}_n$  the colors need to strictly decrease,  $e_{\lambda^{\mathbf{AA}}(\sigma)}(\mathbf{x})$  enumerates the colored permutations with colors in  $S = \mathbb{P}$  and underlying uncolored permutation  $\sigma$ .

**Theorem 4.3** (Theorem 1.4). *We have*

$$(4.1) \quad \left( \sum_{n \geq 0} (-1)^n h_n(\mathbf{x}) \frac{y^n}{n!} \right)^{-1} = \sum_{n \geq 0} \left( \sum_{\sigma \in \mathfrak{S}_n} e_{\lambda(\sigma)}(\mathbf{x}) \right) \frac{y^n}{n!},$$

where  $\mathfrak{S}_n$  is the set of permutations of  $[n] =: \{1, 2, \dots, n\}$  and  $\lambda(\sigma)$  is the ascending adjacent type of  $\sigma \in \mathfrak{S}_n$ .

*Proof.* Let  $\mathcal{A}_S$  be the alphabet with colored letters  $a^c$  where  $a \in [n]$  and  $c \in \mathbb{P}$ . And consider the set of forbidden links  $\overline{\mathcal{L}(\mathcal{A}_S)}$  to be of the form  $a^{c_1} b^{c_2}$  with  $a < b$  and  $c_1 \leq c_2$ . The forbidden words are of the form  $1^{c_1} 2^{c_2} \dots n^{c_n}$  with  $c_1 \leq c_2 \leq \dots \leq c_n$ . Since each of the forbidden colored words is completely determined after selecting a multiset of colors, the generating polynomial of the forbidden words is  $h_n(\mathbf{x})$  and

$$\overline{F}(y) = \sum_{n \geq 0} (-1)^n h_n(\mathbf{x}) \frac{y^n}{n!}.$$

The allowed words are colored permutations whose generating function corresponds by the comments above to the right-hand side of equation (4.1). Applying Theorem 4.1 completes the proof of this theorem.  $\square$

## 5. SPECIALIZATIONS

A *composition*  $\nu$  is a weak composition such that  $\nu(i) \neq 0$  for all  $i \leq \ell(\nu)$ . In other words, a composition can be thought as a finite weak composition with strictly positive parts. For example  $(3, 2, 1, 1, 2, 1)$  is a composition of 10. We denote  $\text{comp}_n$  the set of compositions of  $n$ . To every composition  $\nu \in \text{comp}_n$  we can associate a partition  $\lambda(\nu)$  as the nonincreasing rearrangement of the parts of  $\nu$ . For example,  $\lambda(3, 2, 1, 1, 2, 1) = (3, 2, 2, 1, 1, 1)$ .

The following Proposition is a special case of a more general Theorem of Egecioglu and Remmel in [5].



**Proposition 5.1** (c.f. [5, Theorem 2.3]). *For every  $n \geq 0$ ,*

$$h_n(\mathbf{x}) = (-1)^n \sum_{\nu \in \text{comp}_n} (-1)^{\ell(\lambda(\nu))} e_{\lambda(\nu)}(\mathbf{x})$$

*Proof.* The proof goes by induction on  $n$ . The cases  $n = 0$  and  $n = 1$  are trivially verified. Denote  $\nu_{\text{last}} := \nu(\ell(\nu))$  the last part of  $\nu$ . Then

$$\begin{aligned} \sum_{\nu \in \text{comp}_n} (-1)^{\ell(\lambda(\nu))} e_{\lambda(\nu)}(\mathbf{x}) &= \sum_{k=1}^n \sum_{\substack{\nu \in \text{comp}_n \\ \nu_{\text{last}}=k}} (-1)^{\ell(\lambda(\nu))} e_{\lambda(\nu)}(\mathbf{x}) \\ &= - \sum_{k=1}^n e_k(\mathbf{x}) \sum_{\nu \in \text{comp}_{n-k}} (-1)^{\ell(\lambda(\nu))} e_{\lambda(\nu)}(\mathbf{x}) \\ &= - \sum_{k=1}^n (-1)^{n-k} e_k(\mathbf{x}) h_{n-k}(\mathbf{x}) \\ &= (-1)^n h_n(\mathbf{x}), \end{aligned}$$

where the last two steps make use of the inductive step and the known identity

$$\sum_{k=0}^n (-1)^{n-k} e_k(\mathbf{x}) h_{n-k}(\mathbf{x}) = \delta_{n,0},$$

that is equivalent to Proposition 1.1.  $\square$

Let  $E : \Lambda \rightarrow \mathbb{Q}[t]$  be the map defined by  $E(e_i(\mathbf{x})) = t$  for all  $i \geq 1$  and  $E(1) = 1$ . Since  $E$  is defined on the generators  $e_i$  it is immediate to check that  $E$  is an algebra homomorphism or *specialization*. In the same manner it is also easy to verify that the specialization  $E$  extends to a specialization  $\tilde{E} : \Lambda[[y]] \rightarrow \mathbb{Q}[t][[y]]$  in the algebra of power series in  $y$  with symmetric function coefficients in  $\Lambda$  (with variables  $\mathbf{x}$ ) defined by applying  $E$  coefficientwise. Moreover, it is also true and easy to verify that  $\tilde{E}$  is a monoid homomorphism  $(\Lambda[[y]], \circ) \rightarrow (\mathbb{Q}[[y]], \circ)$ , where  $\circ$  indicates composition of power series, i.e., for power series  $f, g \in \Lambda[[y]]$   $\tilde{E}(f[g(y)]) = \tilde{E}(f)[\tilde{E}(g)(y)]$ . Note that for any  $\lambda \vdash n$  we have that

$$(5.1) \quad E(e_\lambda(\mathbf{x})) = t^{\ell(\lambda)}.$$

**Lemma 5.2.** *For every  $n \geq 1$ ,*

$$(5.2) \quad E(h_n(\mathbf{x})) = t(t-1)^{n-1}.$$

*Proof.* Using Proposition 5.1 and equation (5.1),

$$\begin{aligned}
E(h_n(\mathbf{x})) &= E \left( (-1)^n \sum_{\nu \in \text{comp}_n} (-1)^{\ell(\lambda(\nu))} e_{\lambda(\nu)}(\mathbf{x}) \right) \\
&= (-1)^n \sum_{\nu \in \text{comp}_n} (-t)^{\ell(\lambda(\nu))} \\
&= (-1)^n \sum_{k=1}^n \binom{n-1}{k-1} (-t)^k \\
&= (-1)^n (-t) (1-t)^{n-1},
\end{aligned}$$

where  $\binom{n-1}{k-1}$  is the number of compositions of  $n$  into  $k$  parts.  $\square$

Applying the specialization  $E$  to definition (3.4), using equation (5.1) and applying the observation in Section 3 that  $\ell(\lambda^{\text{AA}}(\theta)) = \text{des}(\theta)$ , we obtain

$$\begin{aligned}
E(SP_n^{(r)}(\mathbf{x})) &= E \left( \sum_{\theta \in \mathcal{Q}_n(r)} e_{\lambda^{\text{AA}}(\theta)}(\mathbf{x}) \right) \\
&= \sum_{\theta \in \mathcal{Q}_n(r)} t^{\ell(\lambda^{\text{AA}}(\theta))} \\
&= \sum_{\theta \in \mathcal{Q}_n(r)} t^{\text{des}(\theta)} \\
&:= A_n^{(r)}(t),
\end{aligned}$$

the  $r$ -th order Eulerian polynomial.

Applying the specialization  $\tilde{E}$  to equation (3.3) and using equations (5.1) and (5.2) we obtain as a corollary Theorem 1.7.

**Theorem 5.3** (Theorem 1.7). *We have*

$$\left( \frac{(1-t)y + (1 - e^{y(1-t)})t}{(1-t)^2} \right)^{\langle -1 \rangle} = \sum_{n \geq 1} A_{n-1}^{(2)}(t) \frac{y^n}{n!}.$$

In the same manner, applying  $\tilde{E}$  to equation (4.1) we obtain as a corollary Riordan's result (Theorem 1.6).

## 6. CONNECTIONS AND FUTURE DIRECTIONS

In this section we discuss some instances where the functions  $SP_n^{(r)}(\mathbf{x})$  appear for the cases  $r = 1$  and  $r = 2$ .

**6.1. Multiplicative inverse and Lagrange inversion.** Propositions 1.1 and 1.2, and Theorems 1.4 and 1.5 are closely related to the problem of finding multiplicative and compositional inverses of general exponential generating functions.

We denote by  $\widetilde{SP}_n^{(r)}(h_1, h_2, \dots)$  the symmetric function  $SP_n^{(r)}(\mathbf{x})$  written as a polynomial in the generators  $h_i(\mathbf{x})$ , i.e.,  $\widetilde{SP}_n^{(r)} \in R[h_1, h_2, \dots]$ . Let

$$(6.1) \quad F(y) = \sum_{n \geq 0} f_n \frac{y^n}{n!},$$

where the  $f_n$  are in some commutative ring  $A$ .

Recall that the multiplicative inverse  $F^{-1}$  exists if and only if  $f_0$  is a unit in  $A$ . The compositional inverse  $F^{(-1)}$  exists if and only if  $f_0 = 0$  and  $f_1$  is a unit in  $A$ . It is easy to check that in the former case, when we apply the specialization  $h_n \mapsto f_n/f_0$  for  $n \geq 0$  to equation (1.1) we obtain that the  $n$ -th coefficient in the power series  $F^{-1}$  is

$$(6.2) \quad n![x^n]F^{-1}(y) = (-1)^n f_0^{-1} \widetilde{SP}_n^{(1)}(f_1/f_0, f_2/f_0, \dots).$$

In the latter case, after applying the specialization  $h_n \mapsto f_{n+1}/f_1$  for  $n \geq 1$  to equation (1.2) we obtain that the  $n$ -th coefficient of  $F^{(-1)}$  is

$$(6.3) \quad n![x^n]F^{(-1)}(y) = (-1)^{n-1} f_1^{-n} \widetilde{SP}_{n-1}^{(2)}(f_2/f_1, f_3/f_1, \dots).$$

**6.2. Poset (co)homology.** The symmetric functions  $SP_n^{(1)}(\mathbf{x})$  and  $SP_n^{(2)}(\mathbf{x})$  also appear in the context of poset topology.

A weighted partition of  $[n]$  is a set  $\{B_1^{\nu_1}, B_2^{\nu_2}, \dots, B_t^{\nu_t}\}$  where  $\{B_1, B_2, \dots, B_t\}$  is a set partition of  $[n]$  and  $\nu_i \in \text{wcomp}_{|B_i|-1}$  for all  $i$ . For  $\nu, \mu \in \text{wcomp}$  we say that  $\nu \leq \mu$  if  $\nu(i) \leq \mu(i)$  for every  $i$ . The *poset of weighted partitions*  $\Pi_n^w$  is the set of weighted partitions of  $[n]$  with order relation given by  $\{A_1^{w_1}, A_2^{w_2}, \dots, A_s^{w_s}\} \leq \{B_1^{v_1}, B_2^{v_2}, \dots, B_t^{v_t}\}$  if the following conditions hold:

- $\{A_1, A_2, \dots, A_s\}$  is a refinement of  $\{B_1, B_2, \dots, B_t\}$  and,
- If  $B_j = A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_l}$  then  $w_{i_1} + w_{i_2} + \dots + w_{i_l} \leq v_j$ .

The poset  $\Pi_n^w$  has one minimal element  $\hat{0} := \{1^{\mathbf{0}}, 2^{\mathbf{0}}, \dots, n^{\mathbf{0}}\}$ , where  $\mathbf{0} = (0, 0, \dots)$ , and maximal elements  $[n]^\mu := \{[n]^\mu\}$  indexed by weak compositions  $\mu \in \text{wcomp}_{n-1}$ . The following result was found in [9] using poset topology techniques (see [9] for all the definitions).

**Theorem 6.1** ([9]). *For all  $n \geq 1$ ,*

$$\sum_{\mu \in \text{wcomp}_{n-1}} \mu_{\Pi_n^w}(\hat{0}, [n]^\mu) \mathbf{x}^\mu = (-1)^{n-1} SP_{n-1}^{(2)}(\mathbf{x}),$$

where  $\mu_{\Pi_n^w}((\hat{0}, [n]^\mu))$  is the Möbius invariant of the maximal interval  $(\hat{0}, [n]^\mu)$ .

It was also proved in [9] that there is an  $\mathfrak{S}_n$ -module isomorphism

$$\mathcal{L}ie(\mu) \simeq_{\mathfrak{S}_n} \tilde{H}^{n-3}((\hat{0}, [n]^\mu)) \otimes \text{sgn}_n$$

for all  $\mu \in \text{wcomp}_{n-1}$ , where  $\mathcal{L}ie(\mu)$  is the multilinear component determined by  $\mu$  of the free Lie algebra with multiple compatible brackets,  $\tilde{H}^{n-3}((\hat{0}, [n]^\mu))$  is the reduced cohomology of the interval  $(\hat{0}, [n]^\mu)$  and  $\text{sgn}_n$  is the sign representation of the symmetric group  $\mathfrak{S}_n$ . The following theorem can be obtained as a corollary of this isomorphism and Theorem 6.1.

**Theorem 6.2** ([9]). *For all  $n \geq 1$ ,*

$$\sum_{\mu \in \text{wcomp}_{n-1}} \dim \mathcal{L}ie(\mu) \mathbf{x}^\mu = SP_{n-1}^{(2)}(\mathbf{x}),$$

where  $\dim V$  is the dimension of the vector space  $V$ .

It turns out that  $SP_{n-1}^{(1)}(\mathbf{x})$  also makes an appearance in the context of poset topology. Here we outline the result and the details will be published in a future article. Let  $B_n$  be the boolean algebra (the poset of subsets of  $[n]$  ordered by inclusion) and  $WC_n$  be the poset of weak compositions of  $n$  with the order relation defined before. Both posets are ranked and hence have well-defined poset maps  $rk : B_n \rightarrow C_{n+1}$  and  $rk : WC_n \rightarrow C_{n+1}$  to the  $n+1$  chain  $C_{n+1}$ . Recall that the *Segre or fiber product*  $A_{f,g}^\times B$  of two poset maps  $f : A \rightarrow C$  and  $g : B \rightarrow C$  is the induced subposet of the product  $A \times B$  with elements  $\{(a, b) \mid f(a) = g(b)\}$ . Denote by  $B_n^w = B_{n_{rk}, rk}^\times WC_n$ , the poset of weighted subsets of  $[n]$ . This poset has one minimal element  $\hat{0} := \emptyset^{\mathbf{0}}$  and maximal elements  $[n]^\mu := \{[n]^\mu\}$  indexed by weak compositions  $\mu \in \text{wcomp}_n$ .

The following theorem can be proven using similar techniques as the ones in [9].

**Theorem 6.3.** *For all  $n \geq 0$ ,*

$$\sum_{\mu \in \text{wcomp}_n} \mu_{B_n^w}((\hat{0}, [n]^\mu)) \mathbf{x}^\mu = (-1)^n SP_n^{(1)}(\mathbf{x}).$$

*Remark 6.4.* Both Theorem 1.5 and Theorem 1.4 can also be derived from Theorem 6.1 and 6.3 using the recursive definition of the Möbius invariant. In particular, Theorem 1.5 was first proved in [9] using this idea.

**6.3. Stable  $n$ -pointed curves.** In [15] another surprising connection with the symmetric functions  $SP_n^{(2)}(\mathbf{x})$  appeared in the context of moduli spaces  $\overline{M}_n$  of stable  $n$ -pointed curves of genus 0. See [15] for the proper definitions and notation. Let  $\omega_n(i) \in H^{2i}(\overline{M}_n, \mathbb{Q})$  denote the Mumford classes and for a partition  $\lambda = 1^{m_1} 2^{m_2} \dots$  denote  $\omega_n^\lambda = \prod_i \omega_n(i)^{m_i}$ . The *higher Weil-Petersson volumes* are defined as

$$WP(\lambda) = \int_{\overline{M}_n} \omega_n^\lambda.$$

The following theorem follows directly from Theorem 1.5 and [15, Theorem 2.4] after making the identifications  $s_k \mapsto -\frac{p_k(\mathbf{x})}{k}$  and  $y \mapsto -y$  ( $s_k$  a formal variable and  $p_k(\mathbf{x})$  the power sum symmetric function) and writing down the proper definitions.

**Theorem 6.5.** *For  $n \geq 0$*

$$SP_n^{(2)}(\mathbf{x}) = \sum_{\lambda \vdash n} (-1)^{n-1-l(\lambda)} WP(\lambda) \frac{p_\lambda(\mathbf{x})}{z_\lambda}$$

In [15] they also provide a recursive definition, a differential equation and the following closed formula for the coefficients  $WP(\lambda)$ . For partitions  $\nu^1, \nu^2, \dots, \nu^k$  denote by  $\nu^1 + \nu^2 + \dots + \nu^k$  the partition obtained by taking the union of all the parts of all the  $\nu^i$ .

**Theorem 6.6** ([15] Corollary 2.3). *For  $n \geq 0$  and  $\lambda \vdash n$*

$$WP(\lambda) = n! \sum_{k=0}^{l(\lambda)} (-1)^{l(\lambda)-k} \binom{n+k}{k} \sum_{\substack{\lambda = \nu^1 + \nu^2 + \dots + \nu^k \\ \nu^i \neq 0}} \frac{\prod_{j=1}^{l(\lambda)} \binom{m_j(\lambda)}{m_j(\nu^1), m_j(\nu^2), \dots, m_j(\nu^k)}}{\prod_{i=1}^k (|\nu^i| + 1)!}.$$

**6.4. Open questions.** Theorems 1.5 and 1.4 say that the symmetric functions  $SP_n^{(1)}(\mathbf{x})$  and  $SP_n^{(2)}(\mathbf{x})$  are involved in the computation of multiplicative and compositional inverses of power series. They are also the generating functions for the Möbius invariants of the maximal intervals of the posets  $B_n^w$  (Theorem 6.3) and  $\Pi_n^w$  (Theorem 6.1), respectively. The symmetric function  $SP_n^{(2)}(\mathbf{x})$  is the dimension generating function of the multilinear components of the free multibracketed Lie algebra (Theorem 6.2) and the generating function (viewed in the  $p$  basis expansion) of the generalized Weil-Petersson volumes of the moduli space  $\overline{M}_n$  (Theorem 6.5). The question now is to understand whether some or all of these results extend to the more general family of symmetric functions  $SP_n^{(r)}(\mathbf{x})$  for  $r \geq 3$ .

**Question 6.7.** *Is there a more general family of posets  $P(n, r)$  such that the weighted generating function for the Möbius invariant of maximal intervals is given up to sign by  $SP_n^{(r)}(\mathbf{x})$  as in Theorems 6.3 and 6.1?*

**Question 6.8.** *Is there any combinatorial context where the functions  $SP_n^{(r)}(\mathbf{x})$  are meaningful for any  $r \geq 3$ ? Are there formulas similar to equations (1.1) and (1.2) for the  $SP_n^{(r)}(\mathbf{x})$  when  $r \geq 3$ ?*

As of the time of the construction of this article we do not know of any partial result in any of these two directions. One central question is whether the family of  $r$ -Stirling permutations for  $r \geq 3$  is the right family of multipermutations to extend the results in this work or if another family of multipermutations generalizing both  $\mathfrak{S}_n$  and  $\mathcal{Q}_n$  is needed.

**6.5. Further work.** Theorem 1.5 can be generalized in a different direction by considering families of normalized  $k$ -ary trees that satisfy certain coloring condition. This generalization however does not include Theorem 1.4 as a special case. We present and explore this generalization in a future article.

#### ACKNOWLEDGMENTS

The author is grateful to Michelle Wachs for her guidance during this project and all the valuable discussions. The author also would like to thank Ira Gessel for the valuable discussions and useful comments to improve both the context and the presentation of the results. A great part of this work was done while the author was supported by NSF Grant DMS 1202755.

APPENDIX: SOME EXPANSIONS OF  $SP_n^{(r)}$  FOR  $r = 1$  AND  $r = 2$ 

$$\begin{aligned}
SP_0^{(1)} &= m_\emptyset = e_\emptyset = h_\emptyset = s_\emptyset = p_\emptyset \\
SP_1^{(1)} &= m_{(1)} = e_{(1)} = h_{(1)} = s_{(1)} = p_{(1)} \\
SP_2^{(1)} &= m_{(2)} + 3m_{(1,1)} \\
&= e_{(2)} + e_{(1,1)} \\
&= -h_{(2)} + 2h_{(1,1)} \\
&= s_{(2)} + 2s_{(1,1)} \\
&= -\frac{1}{2}p_{(2)} + \frac{3}{2}p_{(1,1)} \\
SP_3^{(1)} &= m_{(3)} + 7m_{(2,1)} + 19m_{(1,1,1)} \\
&= e_{(3)} + 4e_{(2,1)} + e_{(1,1,1)} \\
&= h_{(3)} - 6h_{(2,1)} + 6h_{(1,1,1)} \\
&= s_{(3)} + 6s_{(2,1)} + 6s_{(1,1,1)} \\
&= \frac{1}{3}p_{(3)} - \frac{5}{2}p_{(2,1)} + \frac{19}{6}p_{(1,1,1)} \\
SP_4^{(1)} &= m_{(4)} + 15m_{(3,1)} + 33m_{(2,2)} + 83m_{(2,1,1)} + 211m_{(1,1,1,1)} \\
&= e_{(4)} + 6e_{(3,1)} + 5e_{(2,2)} + 11e_{(2,1,1)} + e_{(1,1,1,1)} \\
&= -h_{(4)} + 8h_{(3,1)} + 6h_{(2,2)} - 36h_{(2,1,1)} + 24h_{(1,1,1,1)} \\
&= s_{(4)} + 14s_{(3,1)} + 18s_{(2,2)} + 36s_{(2,1,1)} + 24s_{(1,1,1,1)} \\
&= -\frac{1}{4}p_{(4)} + \frac{7}{3}p_{(3,1)} + \frac{11}{8}p_{(2,2)} - \frac{45}{4}p_{(2,1,1)} + \frac{211}{24}p_{(1,1,1,1)} \\
SP_5^{(1)} &= m_{(5)} + 31m_{(4,1)} + 131m_{(3,2)} + 311m_{(3,1,1)} + 621m_{(2,2,1)} + 1501m_{(2,1,1,1)} + 3651m_{(1,1,1,1,1)} \\
&= e_{(5)} + 8e_{(4,1)} + 18e_{(3,2)} + 23e_{(3,1,1)} + 43e_{(2,2,1)} + 26e_{(2,1,1,1)} + e_{(1,1,1,1,1)} \\
&= h_{(5)} - 10h_{(4,1)} - 20h_{(3,2)} + 60h_{(3,1,1)} + 90h_{(2,2,1)} - 240h_{(2,1,1,1)} + 120h_{(1,1,1,1,1)} \\
&= s_{(5)} + 30s_{(4,1)} + 100s_{(3,2)} + 150s_{(3,1,1)} + 210s_{(2,2,1)} + 240s_{(2,1,1,1)} + 120s_{(1,1,1,1,1)} \\
&= \frac{1}{5}p_{(5)} - \frac{9}{4}p_{(4,1)} - \frac{19}{6}p_{(3,2)} + \frac{27}{2}p_{(3,1,1)} + \frac{131}{8}p_{(2,2,1)} - \frac{649}{12}p_{(2,1,1,1)} + \frac{1217}{40}p_{(1,1,1,1,1)} \\
SP_6^{(1)} &= m_{(6)} + 63m_{(5,1)} + 473m_{(4,2)} + 1075m_{(4,1,1)} + 883m_{(3,3)} + 3755m_{(3,2,1)} + 8727m_{(3,1,1,1)} \\
&\quad + 7015m_{(2,2,2)} + 16417m_{(2,2,1,1)} + 38559m_{(2,1,1,1,1)} + 90921m_{(1,1,1,1,1,1)} \\
&= e_{(6)} + 10e_{(5,1)} + 28e_{(4,2)} + 39e_{(4,1,1)} + 19e_{(3,3)} + 202e_{(3,2,1)} + 72e_{(3,1,1,1)} \\
&\quad + 61e_{(2,2,2)} + 230e_{(2,2,1,1)} + 57e_{(2,1,1,1,1)} + e_{(1,1,1,1,1,1)} \\
&= -h_{(6)} + 12h_{(5,1)} + 30h_{(4,2)} - 90h_{(4,1,1)} + 20h_{(3,3)} - 360h_{(3,2,1)} + 480h_{(3,1,1,1)} \\
&\quad - 90h_{(2,2,2)} + 1080h_{(2,2,1,1)} - 1800h_{(2,1,1,1,1)} + 720h_{(1,1,1,1,1,1)} \\
&= s_{(6)} + 62s_{(5,1)} + 410s_{(4,2)} + 540s_{(4,1,1)} + 410s_{(3,3)} + 1860s_{(3,2,1)} + 1560s_{(3,1,1,1)} \\
&\quad + 990s_{(2,2,2)} + 2160s_{(2,2,1,1)} + 1800s_{(2,1,1,1,1)} + 720s_{(1,1,1,1,1,1)} \\
&= -\frac{1}{6}p_{(6)} + \frac{11}{5}p_{(5,1)} + \frac{29}{8}p_{(4,2)} - \frac{127}{8}p_{(4,1,1)} + \frac{13}{6}p_{(3,3)} - \frac{93}{2}p_{(3,2,1)} + \frac{475}{6}p_{(3,1,1,1)} \\
&\quad - \frac{451}{48}p_{(2,2,2)} + \frac{2353}{16}p_{(2,2,1,1)} - \frac{4601}{16}p_{(2,1,1,1,1)} + \frac{30307}{240}p_{(1,1,1,1,1,1)}
\end{aligned}$$

TABLE 2. Expansion of  $SP_n^{(1)}$  in different bases for  $n = 0, \dots, 6$

$$\begin{aligned}
SP_0^{(2)} &= m_\emptyset = e_\emptyset = h_\emptyset = s_\emptyset = p_\emptyset \\
SP_1^{(2)} &= m_{(1)} = e_{(1)} = h_{(1)} = s_{(1)} = p_{(1)} \\
SP_2^{(2)} &= 2m_{(2)} + 5m_{(1,1)} \\
&= e_{(2)} + 2e_{(1,1)} \\
&= -h_{(2)} + 3h_{(1,1)} \\
&= 2s_{(2)} + 3s_{(1,1)} \\
&= -\frac{1}{2}p_{(2)} + \frac{5}{2}p_{(1,1)} \\
SP_3^{(2)} &= 6m_{(3)} + 26m_{(2,1)} + 61m_{(1,1,1)} \\
&= e_{(3)} + 8e_{(2,1)} + 6e_{(1,1,1)} \\
&= h_{(3)} - 10h_{(2,1)} + 15h_{(1,1,1)} \\
&= 6s_{(3)} + 20s_{(2,1)} + 15s_{(1,1,1)} \\
&= \frac{1}{3}p_{(3)} - \frac{9}{2}p_{(2,1)} + \frac{61}{6}p_{(1,1,1)} \\
SP_4^{(2)} &= 24m_{(4)} + 154m_{(3,1)} + 269m_{(2,2)} + 609m_{(2,1,1)} + 1379m_{(1,1,1,1)} \\
&= e_{(4)} + 13e_{(3,1)} + 9e_{(2,2)} + 58e_{(2,1,1)} + 24e_{(1,1,1,1)} \\
&= -h_{(4)} + 15h_{(3,1)} + 10h_{(2,2)} - 105h_{(2,1,1)} + 105h_{(1,1,1,1)} \\
&= 24s_{(4)} + 130s_{(3,1)} + 115s_{(2,2)} + 210s_{(2,1,1)} + 105s_{(1,1,1,1)} \\
&= -\frac{1}{4}p_{(4)} + \frac{14}{3}p_{(3,1)} + \frac{19}{8}p_{(2,2)} - \frac{161}{4}p_{(2,1,1)} + \frac{1379}{24}p_{(1,1,1,1)} \\
SP_5^{(2)} &= 120m_{(5)} + 1044m_{(4,1)} + 2724m_{(3,2)} + 6028m_{(3,1,1)} + 10193m_{(2,2,1)} + 22562m_{(2,1,1,1)} + 49946m_{(1,1,1,1,1)} \\
&= e_{(5)} + 19e_{(4,1)} + 33e_{(3,2)} + 136e_{(3,1,1)} + 192e_{(2,2,1)} + 444e_{(2,1,1,1)} + 120e_{(1,1,1,1,1)} \\
&= h_{(5)} - 21h_{(4,1)} - 35h_{(3,2)} + 210h_{(3,1,1)} + 280h_{(2,2,1)} - 1260h_{(2,1,1,1)} + 945h_{(1,1,1,1,1)} \\
&= 120s_{(5)} + 924s_{(4,1)} + 1680s_{(3,2)} + 2380s_{(3,1,1)} + 2485s_{(2,2,1)} + 2520s_{(2,1,1,1)} + 945s_{(1,1,1,1,1)} \\
&= \frac{1}{5}p_{(5)} - 5p_{(4,1)} - \frac{17}{3}p_{(3,2)} + \frac{172}{3}p_{(3,1,1)} + \frac{235}{4}p_{(2,2,1)} - \frac{2411}{6}p_{(2,1,1,1)} + \frac{24973}{60}p_{(1,1,1,1,1)} \\
SP_6^{(2)} &= 720m_{(6)} + 8028m_{(5,1)} + 28636m_{(4,2)} + 62376m_{(4,1,1)} + 42881m_{(3,3)} + 154629m_{(3,2,1)} + 336909m_{(3,1,1,1)} \\
&\quad + 255982m_{(2,2,2)} + 557787m_{(2,2,1,1)} + 1215507m_{(2,1,1,1,1)} + 2648967m_{(1,1,1,1,1,1)} \\
&= e_{(6)} + 26e_{(5,1)} + 54e_{(4,2)} + 269e_{(4,1,1)} + 34e_{(3,3)} + 958e_{(3,2,1)} + 1396e_{(3,1,1,1)} \\
&\quad + 225e_{(2,2,2)} + 3004e_{(2,2,1,1)} + 3708e_{(2,1,1,1,1)} + 720e_{(1,1,1,1,1,1)} \\
&= -h_{(6)} + 28h_{(5,1)} + 56h_{(4,2)} - 378h_{(4,1,1)} + 35h_{(3,3)} - 1260h_{(3,2,1)} + 3150h_{(3,1,1,1)} \\
&\quad - 280h_{(2,2,2)} + 6300h_{(2,2,1,1)} - 17325h_{(2,1,1,1,1)} + 10395h_{(1,1,1,1,1,1)} \\
&= 720s_{(6)} + 7308s_{(5,1)} + 20608s_{(4,2)} + 26432s_{(4,1,1)} + 14245s_{(3,3)} + 57400s_{(3,2,1)} + 44100s_{(3,1,1,1)} \\
&\quad + 23345s_{(2,2,2)} + 47880s_{(2,2,1,1)} + 34650s_{(2,1,1,1,1)} + 10395s_{(1,1,1,1,1,1)} \\
&= -\frac{1}{6}p_{(6)} + \frac{27}{5}p_{(5,1)} + \frac{55}{8}p_{(4,2)} - \frac{645}{8}p_{(4,1,1)} + \frac{23}{6}p_{(3,3)} - \frac{369}{2}p_{(3,2,1)} + \frac{4391}{6}p_{(3,1,1,1)} \\
&\quad - \frac{1513}{48}p_{(2,2,2)} + \frac{18087}{16}p_{(2,2,1,1)} - \frac{72651}{16}p_{(2,1,1,1,1)} + \frac{882989}{240}p_{(1,1,1,1,1,1)}
\end{aligned}$$

TABLE 3. Expansion of  $SP_n^{(2)}$  in different bases for  $n = 0, \dots, 6$



## REFERENCES

- [1] M. Bóna. Real zeros and normal distribution for statistics on Stirling permutations defined by Gessel and Stanley. *SIAM J. Discrete Math.*, 23(1):401–406, 2008/09.
- [2] L. Carlitz, R. Scoville, and T. Vaughan. Enumeration of pairs of sequences by rises, falls and levels. *Manuscripta Math.*, 19(3):211–243, 1976.
- [3] V. Dotsenko. Pattern avoidance in labelled trees. *Sém. Lothar. Combin.*, 67:Art. B67b, 27, 2011/12.
- [4] B. Drake. *An inversion theorem for labeled trees and some limits of areas under lattice paths*. ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)—Brandeis University.
- [5] Ö. Eğecioğlu and J. B. Remmel. Brick tabloids and the connection matrices between bases of symmetric functions. *Discrete Appl. Math.*, 34(1-3):107–120, 1991. *Combinatorics and theoretical computer science* (Washington, DC, 1989).
- [6] R. Fröberg. Determination of a class of Poincaré series. *Math. Scand.*, 37(1):29–39, 1975.
- [7] I. Gessel and R. P. Stanley. Stirling polynomials. *J. Combinatorial Theory Ser. A*, 24(1):24–33, 1978.
- [8] I. M. Gessel. *Generating functions and enumeration of sequences*. ProQuest LLC, Ann Arbor, MI, 1977. Thesis (Ph.D.)—Massachusetts Institute of Technology.
- [9] R. S. González D’León. On the free Lie algebra with multiple brackets. *ArXiv e-prints*, Aug. 2014.
- [10] R. S. González D’León and M. L. Wachs. On the (co)homology of the poset of weighted partitions. *ArXiv e-prints*, Sept. 2013.
- [11] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete mathematics: a foundation for computer science*, volume 2. Addison-Wesley Reading, MA, 1994.
- [12] J. Haglund and M. Visontai. Stable multivariate Eulerian polynomials and generalized Stirling permutations. *European J. Combin.*, 33(4):477–487, 2012.
- [13] M. D. Haiman. Conjectures on the quotient ring by diagonal invariants. *J. Algebraic Combin.*, 3(1):17–76, 1994.
- [14] S. Janson, M. Kuba, and A. Panholzer. Generalized Stirling permutations, families of increasing trees and urn models. *J. Combin. Theory Ser. A*, 118(1):94–114, 2011.
- [15] R. Kaufmann, Y. Manin, and D. Zagier. Higher Weil-Petersson volumes of moduli spaces of stable  $n$ -pointed curves. *Comm. Math. Phys.*, 181(3):763–787, 1996.
- [16] M. Kuba and A. Panholzer. Analysis of statistics for generalized Stirling permutations. *Combin. Probab. Comput.*, 20(6):875–910, 2011.
- [17] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.
- [18] S. Park. Inverse descents of  $r$ -multipermutations. *Discrete Math.*, 132(1-3):215–229, 1994.

- [19] S. Park.  $P$ -partitions and  $q$ -Stirling numbers. *J. Combin. Theory Ser. A*, 68(1):33–52, 1994.
- [20] S. Park. The  $r$ -multi-permutations. *J. Combin. Theory Ser. A*, 67(1):44–71, 1994.
- [21] J. Riordan. Triangular permutation numbers. *Proc. Amer. Math. Soc.*, 2:429–432, 1951.
- [22] R. P. Stanley. Log-concave and unimodal sequences in algebra, combinatorics, and geometry. In *Graph theory and its applications: East and West (Jinan, 1986)*, volume 576 of *Ann. New York Acad. Sci.*, pages 500–535. New York Acad. Sci., New York, 1989.
- [23] R. P. Stanley. Parking functions and noncrossing partitions. *Electron. J. Combin.*, 4(2):Research Paper 20, approx. 14 pp. (electronic), 1997. The Wilf Festschrift (Philadelphia, PA, 1996).
- [24] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.
- [25] R. P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.
- [26] M. L. Wachs. On the (co)homology of the partition lattice and the free Lie algebra. *Discrete Math.*, 193(1-3):287–319, 1998. Selected papers in honor of Adriano Garsia (Taormina, 1994).